Solve the differential equation \( x^2y'' + 3xy' + (1 + x)y = 0 \) about \( x = 0 \).

Comparing the differential equation with \( P(x)y'' + Q(x)y' + R(x)y = 0 \), we get \( P(x) = x^2 \). Note that \( x = 0 \) makes \( P(x) \) zero, so \( x = 0 \) is a singular point.

Since \( \lim_{x \to 0} \frac{Q(x)}{P(x)} = \lim_{x \to 0} \frac{3x}{x^2} = 3 \) and \( \lim_{x \to 0} \left( x - 0 \right)^2 \frac{R(x)}{P(x)} = \lim_{x \to 0} x^2 \frac{1 + x}{x^2} = 1 \) we conclude that \( x = 0 \) is a regular singular point.

Let \( y = x^r \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+r} \) be a solution.

Then \( y' = \sum_{n=0}^{\infty} (n+r)a_n x^{n+r-1} \), \( y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1)a_n x^{n+r-2} \).

Substituting these into the differential equation \( x^2y'' + 3xy' + y + xy = 0 \), we get

\[
\sum_{n=0}^{\infty} (n+r)(n+r-1)a_n x^{n+r} + 3 \sum_{n=0}^{\infty} (n+r)a_n x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r+1} = 0
\]

\[
[r(r-1) + 3r + 1]a_0 x^r + \sum_{n=1}^{\infty} \{ [(n+r)(n+r-1) + 3(n+r) + 1]a_n + a_{n-1} \} x^{n+r} = 0
\]

\[
F(r)a_0 x^r + \sum_{n=1}^{\infty} \{ F(n+r)a_n + a_{n-1} \} x^{n+r} = 0, \quad \text{where} \quad F(r) := r(r-1) + 3r + 1 = (r+1)^2
\]

\[(a_0 \neq 0) \implies F(r) = 0, \quad \text{and} \quad F(n+r)a_n + a_{n-1} = 0, \quad n \geq 1\]

\[\implies r = -1, -1 \quad a_n(r) = -\frac{a_{n-1}(r)}{F(n+r)}, \quad n \geq 1 \quad \text{(Recurrence Relation)}\]

From the recurrence relation, we get

\[a_1 = -\frac{a_0}{F(r+1)} = -\frac{a_0}{(r+2)^2}\]

\[a_2 = -\frac{a_1}{F(r+2)} = \frac{a_0}{(r+3)^2(r+2)^2}\]

\[a_3 = -\frac{a_2}{F(r+3)} = -\frac{a_0}{(r+4)^2(r+3)^2(r+2)^2}\]

\[\vdots\]

\[a_n(r) = (-1)^n \frac{a_0}{(r+n+1)^2(r+n)^2 \ldots (r+2)^2}, \quad n \geq 1\]
For the first solution \((\forall r = -1)\), we get
\[
a_n = (-1)^n \frac{1}{n^2 \cdot (n-1)^2 \cdots 1^2} = (-1)^n \frac{1}{(n!)^2}, \quad n \geq 0, \quad (a_0 = 1).
\]

Thus the first solution is
\[
y(x) = y_1(x) = \sum_{n=0}^{\infty} a_n x^{n-1} = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(n!)^2} x^{n-1}, \quad x > 0.
\]

Since \(r_1 = r_2\), the second solution has the form
\[
y = y_2(x) = y_1(x) \ln x + x^{-1} \sum_{n=1}^{\infty} a'_n(r_1)x^n, \quad x > 0.
\]

To find \(a'_n(r_1)\) we write \(a_n(r) = (-1)^n \frac{1}{G_n(r)}\) taking \(a_0 = 1\) where \(G_n(r) := (r + n + 1)^2(r + n)^2 \cdots (r + 2)^2\).

Note that
\[
\frac{G'_n(r)}{G_n(r)} = \frac{2}{r + n + 1} + \frac{2}{r + n} + \ldots + \frac{2}{r + 2}, \text{ using the fact that if}
\]
\[
f(x) = (x - a_1)^{b_1} (x - a_2)^{b_2} \ldots (x - a_n)^{b_n}, \text{ then } f'(x) f(x) = \frac{b_1}{x - a_1} + \frac{b_2}{x - a_2} + \ldots + \frac{b_n}{x - a_n}.
\]

Now
\[
a_n(r_1) = \frac{d}{dr} \left[ a_n(r) \right]_{r=r_1}
\]
\[
= \frac{d}{dr} \left[ (-1)^n \frac{1}{G_n(r)} \right]_{r=-1} = (-1)^n \cdot \frac{1}{G'_n(-1)} \cdot G_n(r) |_{r=-1} = (-1)^{n+1} \frac{G'_n(r)}{G_n(r)} \cdot \frac{1}{G_n(r)} |_{r=-1}
\]
\[
= (-1)^{n+1} \left[ \frac{2}{r + n + 1} + \frac{2}{r + n} + \ldots + \frac{2}{r + 2} \right] \left[ \frac{1}{(r + n + 1)^2(r + n)^2 \cdots (r + 2)^2} \right]_{r=-1}
\]
\[
= (-1)^{n+1} \left[ \frac{2}{n} + \frac{2}{n-1} + \ldots + \frac{2}{1} \right] \left[ \frac{1}{n^2(n-1)^2 \cdots (1)^2} \right]
\]
\[
= (-1)^{n+1} 2H_n \frac{1}{(n!)^2}, \quad \text{where } H_n := 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}
\]

Thus a second solution is
\[
y_2(x) = y_1(x) \ln x + x^{-1} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2H_n}{(n!)^2} x^n, \quad x > 0,
\]
where \(y_1\) is the first solution.