Factor the following matrix A as $A = QR$ where the columns of Q are orthonormal vectors found by the columns of A.

$$A = \begin{bmatrix} 3 & 1 \\ 6 & 2 \\ 0 & 2 \end{bmatrix}$$

Solution: Let's denote the columns of A as the vectors

$$\vec{x_1} = \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix}, \quad \vec{x_2} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

Then first we find an orthogonal set $\{\vec{v_1}, \vec{v_2}\}$ as follows: (Gram-Schimdt process)

$$\vec{v_1} = \vec{x}_1$$

$$\vec{v}_2 = \vec{x}_2 - \left(\frac{\vec{x}_2 \cdot \vec{x}_1}{\vec{x}_1 \cdot \vec{x}_1}\right)\vec{x}_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} - \frac{15}{45}\begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$$

Next we normalize $\{\vec{v_1}, \vec{v_2}\}$ to find an orthonormal set:

$$\vec{u}_1 = \frac{1}{\|\vec{v}_1\|}\vec{v}_1 = \frac{1}{\sqrt{45}}\begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \\ 0 \end{bmatrix}$$

$$\vec{u}_2 = \frac{1}{\|\vec{v}_2\|}\vec{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Note that the set $\{\vec{u_1}, \vec{u_2}\}$ forms the columns of Q. That is,

$$Q = \begin{bmatrix} \frac{1}{\sqrt{5}} & 0 \\ \frac{2}{\sqrt{5}} & 0 \\ 0 & 1 \end{bmatrix}$$

Next we find R as follows:

$$R = Q^T A = \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 3 & 1 \\ 6 & 2 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} \frac{15}{\sqrt{5}} & \frac{5}{\sqrt{5}} \\ 0 & 2 \end{bmatrix}$$

Thus, $A = QR$ where Q and R are found above.