Find an **orthonormal basis** of the subspace spanned by the vectors $\begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$

Solution: Let's denote the given vectors as

$$\overrightarrow{x_1} = \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix}, \quad \overrightarrow{x_2} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}.$$

Then first we find an **orthogonal basis** as $\{\overrightarrow{v_1}, \overrightarrow{v_2}\}$ as follows:

$$\overrightarrow{v_1} = \overrightarrow{x_1}$$

$$\overrightarrow{v}_2 = \overrightarrow{x}_2 - \left(\frac{\overrightarrow{x}_2 \cdot \overrightarrow{x}_1}{\overrightarrow{x}_1 \cdot \overrightarrow{x}_1} \right) \overrightarrow{x}_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} - \frac{15}{45} \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$$

Next we normalize $\{\overrightarrow{v_1}, \overrightarrow{v_2}\}$ to find an **orthonormal** basis:

$$\overrightarrow{u}_1 = \frac{1}{\|\overrightarrow{v}_1\|} \overrightarrow{v}_1 = \frac{1}{\sqrt{45}} \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \\ 0 \end{bmatrix}$$

$$\overrightarrow{u}_2 = \frac{1}{\|\overrightarrow{v}_2\|} \overrightarrow{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Note that the set $\{\overrightarrow{u_1}, \overrightarrow{u_2}\}$ is an **orthonormal** basis.