Partial Fraction Decomposition examples

① $\dfrac{1}{s^2-5s+4}$.

$s^2-5s+4 = (s-1)(s-4)$

Let $\dfrac{1}{s^2-5s+4} = \dfrac{A}{s-1} + \dfrac{B}{s-4}$

$\Rightarrow \quad 1 = A(s-4) + B(s-1)$

$s=4: \quad 1 = 0 + 3B \Rightarrow B = \frac{1}{3}$

$s=1: \quad 1 = -3A + 0 \Rightarrow A = -\frac{1}{3}$

Thus $\dfrac{1}{s^2-5s+4} = \dfrac{-1/3}{s-1} + \dfrac{1/3}{s-4}$.

② $\dfrac{s-2}{s^2-2s-15}$.

$s^2-2s-15 = (s-5)(s+3)$

Let $\dfrac{s-2}{s^2-2s-15} = \dfrac{A}{s-5} + \dfrac{B}{s+3}$

$\Rightarrow \quad s-2 = A(s+3) + B(s-5)$

$s=5: \quad 3 = 8A + 0 \Rightarrow A = \frac{3}{8}$

$s=-3: \quad -5 = 0 + (-8B) \Rightarrow B = \frac{5}{8}$

Thus $\dfrac{s-2}{s^2-2s-15} = \dfrac{3/8}{s-5} + \dfrac{5/8}{s+3}$.

③ $\dfrac{1}{s(s^2-9)}$.

$s^2-9 = (s+3)(s-3)$

Let $\dfrac{1}{s(s^2-9)} = \dfrac{A}{s} + \dfrac{B}{s+3} + \dfrac{C}{s-3}$] * $s(s+3)(s-3)$

$\Rightarrow \quad 1 = A(s+3)(s-3) + Bs(s-3) + Cs(s+3)$

$s=3: \quad 1 = 0+0+C\cdot 3\cdot 8 \quad \Rightarrow C = \frac{1}{18}$

$s=-3: \quad 1 = 0 + B(-3)\cdot(-6) + 0 \quad \Rightarrow B = \frac{1}{18}$

$s=0: \quad 1 = A\cdot 3\cdot(-3) + 0 + 0 \quad \Rightarrow A = -\frac{1}{9}$

Thus $\dfrac{1}{s(s^2-9)} = \dfrac{-1/9}{s} + \dfrac{1/18}{s+3} + \dfrac{1/18}{s-3}$.

④ $\dfrac{5}{s(s^2+4)}$

Let $\dfrac{5}{s(s^2+4)} = \dfrac{A}{s} + \dfrac{Bs+C}{s^2+4}$

$\Rightarrow \quad 5 = A(s^2+4) + s(Bs+C)$

$5 = As^2 + 4A + Bs^2 + Cs$

$0\cdot s^2 + 0\cdot s + 5 = (A+B)s^2 + Cs + 4A$

(equating the coefficients)

$A+B=0, \quad C=0, \quad 4A=5$

or, $A = \frac{5}{4}$

$A+B=0$

$\frac{5}{4} + B = 0 \quad \Rightarrow B = -\frac{5}{4}$

Thus $\dfrac{5}{s(s^2+4)} = \dfrac{5/4}{s} + \dfrac{-\frac{5}{4}s}{s^2+4}$.

⑤ $\dfrac{3}{(s^2+1)(s^2+9)}$.

Let $\dfrac{3}{(s^2+1)(s^2+9)} = \dfrac{A}{s^2+1} + \dfrac{B}{s^2+9}$

($\dfrac{As+B}{s^2+1}$ etc. not needed as the top does not contain s)

$\Rightarrow \quad 3 = A(s^2+9) + B(s^2+1)$

$0\cdot s^2 + 3 = (A+B)s^2 + (9A+B)$

$\Rightarrow \begin{cases} A+B=0 \\ 9A+B=3 \end{cases}$ solve: $B=-A$.

$9A+(-A)=3$ or, $8A=3$ or, $A=\frac{3}{8}$.

Then $B=-\frac{3}{8}$

Thus $\dfrac{3}{(s^2+1)(s^2+9)} = \dfrac{3/8}{s^2+1} + \dfrac{-3/8}{s^2+9}$.

⑥ $\dfrac{2}{s(s^2+4s+5)}$

Let $\dfrac{2}{s(s^2+4s+5)} = \dfrac{A}{s} + \dfrac{Bs+C}{s^2+4s+5}$

$\Rightarrow \quad 2 = A(s^2+4s+5) + s(Bs+C)$

$0\cdot s^2 + 0\cdot s + 2 = (A+B)s^2 + (4A+C)s + 5A$

$\Rightarrow \begin{cases} A+B=0 \\ 4A+C=0 \\ 5A=2 \end{cases}$ solve: $A=\frac{2}{5}$.

$A+B=0$

$\frac{2}{5}+B=0$ or, $B=-\frac{2}{5}$

$4A+C=0$

$\frac{8}{5}+C=0$ or, $C=-\frac{8}{5}$.

Thus $\dfrac{2}{s(s^2+4s+5)} = \dfrac{2/5}{s} + \dfrac{-\frac{2}{5}s-\frac{8}{5}}{s^2+4s+5}$.

Completing the square:

(1) $s^2+2s+2 = (s^2+2s+1)+1 = (s+1)^2+1$

(2) $s^2-4s+7 = (s^2-4s+4)+3 = (s-2)^2+3$

(3) $s^2+3s+1 = \underbrace{s^2+2\cdot s\cdot \frac{3}{2}+\left(\frac{3}{2}\right)^2}\underbrace{-\left(\frac{3}{2}\right)^2+1}_{-\frac{9}{4}+1}$

$$= \left(s+\frac{3}{2}\right)^2-\frac{5}{4}$$

(4) $s^2-s+1 = s^2-2\cdot s\cdot\frac{1}{2}+\left(\frac{1}{2}\right)^2\underbrace{-\left(\frac{1}{2}\right)^2+1}_{-\frac{1}{4}+1}$

$$= \left(s-\frac{1}{2}\right)^2+\frac{3}{4}$$