Worksheet 9.1 and 9.3

Full Name: ______________________ Score: ________

1. (a) Verify that e^t and e^{2t} are both solutions of the differential equation $y'' - 3y' + 2y = 0$.

$y = e^t$ $\quad (e^t) - 3(e^t) + 2(e^t) = 0 \quad$ $y = e^{2t}$ $\quad (4e^{2t}) - 3(2e^{2t}) + 2(e^{2t}) = 0$

$y' = e^t$ $\quad -2e^t + 2e^t = 0 \quad$ $y' = 2e^{2t}$ $\quad -2e^{2t} + 2e^{2t} = 0$

$y'' = e^t$ $\quad 0 = 0 \quad$ $y'' = 4e^{2t}$ $\quad 0 = 0$

(b) Now, let C_1 and C_2 be any constants. Verify that $C_1e^t + C_2e^{2t}$ is a solution of the differential equation $y'' - 3y' + 2y = 0$.

$y = C_1e^t + C_2e^{2t}$ $\quad (C_1e^t + 4C_2e^{2t}) - 3(C_1e^t + 2C_2e^{2t}) + 2(C_1e^t + C_2e^{2t}) = 0$

$y' = C_1e^t + 2C_2e^{2t}$ $\quad C_1e^t - 3C_1e^t + 2C_1e^t + 4C_2e^{2t} - 6C_2e^{2t} + 2C_2e^{2t} = 0$

$y'' = C_1e^t + 4C_2e^{2t}$ $\quad 0 + 0 = 0$

2. Find the general solutions of the following differential equations.

(a) $\frac{dy}{dx} = \frac{3x^2}{5y^4}$

$$5y^4\,dy = 3x^2\,dx$$

$$\int 5y^4\,dy = \int 3x^2\,dx$$

$$y^5 = x^3 + C$$

$$y = \sqrt[5]{x^3 + C}$$

(b) $\frac{dy}{dx} = \frac{y^2}{x}$

$$\frac{dy}{y^2} = \frac{dx}{x} \qquad \int \frac{1}{y^2}\,dy = \int \frac{1}{x}\,dx$$

$$-\frac{1}{y} = \ln|x| + C$$

$$-1 = y(\ln|x|) + C$$

$$\frac{-1}{\ln|x| + C} = y$$

3. Find the solution that satisfies the given initial condition.

(a) $\frac{dy}{dx} = 2y, \qquad y(0) = 4.$

$\frac{dy}{2y} = dx \qquad \int \frac{dy}{2y} = \int dx$

$\frac{1}{2} \ln|y| = x + C$

$e^{\ln|y|} = e^{2x+C}$

$|y| = e^{2x+C}$

$y = Ce^{2x}$

$4 = Ce^{2\cdot 0}$

$C = 4$

$y = 4e^{2x}$

(b) $\frac{dy}{dx} = x\cos^2 y, \qquad y(1) = 0.$

$\frac{dy}{\cos^2 y} = x\,dx \qquad \int \sec^2 y\, dy = \int x\,dx$

$\tan y = \frac{1}{2}x^2 + C$

$y = \tan^{-1}(\frac{1}{2}x^2 + C)$

$0 = \tan^{-1}(\frac{1}{2}(1)^2 + C)$

$0 = \tan^{-1}(\frac{1}{2} + C)$

$0 = \tan(0) \qquad C = -\frac{1}{2}$

$y = \tan^{-1}(\frac{1}{2}x^2 - \frac{1}{2})$

(c) $\frac{dy}{dx} = -y(y-10), \qquad y(0) = 5$

$\frac{dy}{dx} = -y(y-10)$

$dy = -y(y-10)\,dx$

$\frac{1}{-y(y-10)}\,dy = dx$

$\int \frac{1}{-y(y-10)}\,dy = \int dx$

$\downarrow$

$\frac{A}{-y} + \frac{B}{y-10}$

$A(y-10) + B(-y) = 1$

$Ay - 10A - By = 1$

$y(A-B) - 10A = 1$

$A - B = 0 \qquad -10A = 1$

$A = -\frac{1}{10}$

$-\frac{1}{10} - B = 0$

$B = -\frac{1}{10}$

$\frac{-\frac{1}{10}}{-y} + \frac{-1/10}{y-10}$

$\int \frac{1/10}{y} - \frac{1/10}{y-10}\,dy = \int dx$

$\frac{1}{10}\ln|y| - \frac{1}{10}\ln|y-10| = x + C_1$

$\ln|y| - \ln|y-10| = 10x + C_2$

$\ln\left|\frac{y}{y-10}\right| = 10x + C_2$

$\left|\frac{y}{y-10}\right| = e^{10x + C_2}$

$\left|\frac{y}{y-10}\right| = e^{10x} \cdot e^{C_2}$

$\frac{y}{y-10} = \pm e^{10x} \cdot C_3$

$\frac{y}{y-10} = C_4 \cdot e^{10x}$

$y = C_4 e^{10x} y - 10C_4 e^{10x}$

$y - C_4 e^{10x} y = -10C_4 e^{10x}$

$y(1 - C_4 e^{10x}) = -10C_4 e^{10x}$

$y = \frac{-10C_4 e^{10x}}{1 - C_4 e^{10x}}$

$5 = \frac{-10C_4 e^{10 \cdot 0}}{1 - C_4 e^{10 \cdot 0}}$

$5 = \frac{-10C_4 \cdot 1}{1 - C_4 \cdot 1}$

$5 - 5C_4 = -10C_4$

$5 = -5C_4$

$C_4 = -1$

$y = \frac{-10(-1)e^{10x}}{1 - (-1)e^{10x}}$

$\boxed{y = \frac{10e^{10x}}{1 + e^{10x}}}$

(c) $\dfrac{dy}{dx} = -y(y-10), \qquad y(0) = 5$

$$\frac{dy}{y(y-10)} = -dx$$

$$\frac{1}{y(y-10)} = \frac{A}{y} + \frac{B}{y-10} = \frac{A(y-10)+By}{y(y-10)}$$

So $A(y-10) + By = 1$

$\Rightarrow (A+B)y - 10A = 1$

$$\begin{cases} A+B=0 \\ -10A = 1 \end{cases} \Rightarrow A = -1/10, \; B = 1/10$$

So $\displaystyle\int \frac{dy}{y(y-10)} = -\int dx$ implies that

$$\int \frac{-1/10}{y}\,dy + \int \frac{1/10}{y-10}\,dy = -\int dx$$

$$-\frac{1}{10}\ln|y| + \frac{1}{10}\ln|y-10| = -x + C_1$$

$\Rightarrow \ln|y| - \ln|y-10| = 10x + C_2$ (mult. by -10)

$$\ln\left|\frac{y}{y-10}\right| = 10x + C_2 \Rightarrow \left|\frac{y}{y-10}\right| = e^{10x+C_2} = C_3 e^{10x}$$

$$\Rightarrow \frac{y}{y-10} = \pm C_3 e^{10x} = C e^{10x}$$

Now we solve for C using $y(0)=5$.

$$\frac{5}{5-10} = Ce^{10(0)} \Rightarrow C = -1$$

So $\dfrac{y}{y-10} = -e^{10x} \Rightarrow y = (y-10)\cdot(-e^{10x}) \Rightarrow y = -ye^{10x} + 10e^{10x}$

$$\Rightarrow y + ye^{10x} = 10e^{10x}$$

$$\Rightarrow y(1+e^{10x}) = 10e^{10x}$$

$$\Rightarrow \boxed{y = \frac{10e^{10x}}{1+e^{10x}}}$$