Worksheet 7.4

Full Name: ______________________________ Score: ________

1. Use a partial fraction decomposition to evaluate each of the following indefinite integrals.

(a) $\int \frac{8x+17}{2x+5}\,dx = \int 4 + \frac{-3}{2x+5}\,dx$

long division

$= 4x - 3\cdot\frac{1}{2}\ln|2x+5| + C$

$= 4x - \frac{3}{2}\ln|2x+5| + C$

(b) $\int \frac{1}{\underbrace{x^2-25}_{(x-5)(x+5)}}\,dx = \int \frac{1/10}{x-5} - \frac{1/10}{x+5}\,dx$

$\frac{1}{(x-5)(x+5)} = \frac{\overset{1/10}{A}}{x-5} + \frac{\overset{-1/10}{B}}{x+5}$

$1 = A(x+5) + B(x-5)$

$\underline{x=5}$: $1 = A\cdot 10 + B\cdot 0$

$A = 1/10$

$\underline{x=-5}$: $1 = A\cdot 0 + B(-5-5)$

$1 = -10B$

$-1/10 = B$

$= \frac{1}{10}\int\frac{1}{x-5}\,dx - \frac{1}{10}\int\frac{1}{x+5}\,dx$

$= \frac{1}{10}\ln|x-5| - \frac{1}{10}\ln|x+5| + C$

OR

$= \frac{1}{10}\ln\left|\frac{x-5}{x+5}\right| + C$

(c) $\displaystyle\int \frac{-4x+22}{\underbrace{x^2-6x+8}_{(x-4)(x-2)}}\,dx = \int \frac{3}{x-4} - \frac{7}{x-2}\,dx$

$= 3\ln|x-4| - 7\ln|x-2| + C$

$\dfrac{-4x+22}{x^2-6x+8} = \dfrac{A}{x-4} + \dfrac{B}{x-2}$

$-4x+22 = A(x-2)+B(x-4)$

$\underline{x=2:}$ $\quad -8+22 = A\cdot 0 + B(2-4)$

$14 = -2B$

$B = -7$

$\underline{x=4}$

$-4\cdot 4+22 = A\cdot 2 + B\cdot 0$

$6 = 2A$

$3 = A$

(d) $\displaystyle\int \frac{6x^3-3x^2-4x+7}{2x^2-x-1}\,dx = \int 3x + \frac{-x+7}{\underbrace{2x^2-x-1}_{(2x+1)(x-1)}}\,dx$

$$\begin{array}{r|l} & 3x \\ 2x^2-x-1 & 6x^3-3x^2-4x+7 \\ & -\ (6x^3-3x^2-3x) \\ \hline & 0+0-x+7 \end{array}$$

$= \displaystyle\int 3x - \frac{13/3}{2x+1} + \frac{2}{x-1}\,dx$

$= \frac{3}{2}x^2 - \frac{13}{3}\cdot\frac{1}{2}\ln|2x+1| + 2\ln|x-1| + C$

$= \frac{3}{2}x^2 - \frac{13}{6}\ln|2x+1| - \ln(x-1)^2 + C$

$\dfrac{-x+7}{(2x+1)(x-1)} = \dfrac{A}{2x+1} + \dfrac{B}{x-1}$

$-x+7 = A(x-1) + B(2x+1)$

$\underline{x=1}$

$-1+7 = A\cdot 0 + B\cdot 3$

$6 = 3B$

$B = 2$

$\underline{x=-1/2}$

$-\frac{1}{2}+7 = A\left(-\frac{1}{2}-1\right) + B\cdot 0$

$\frac{13}{2} = A\left(-\frac{3}{2}\right)$

$A = -\frac{13}{3}$

(e) $\int \frac{-3x^2+18x-18}{\underbrace{x^3-9x}_{x(x-3)(x+3)}}\,dx \quad = \int \frac{2}{x} + \frac{1/2}{x-3} - \frac{11/2}{x+3}\,dx$

$= 2\ln|x| + \frac{1}{2}\ln|x-3| - \frac{11}{2}\ln|x+3| + C$

$\frac{-3x^2+18x-18}{x(x-3)(x+3)} = \frac{\overset{2}{\cancel{A}}}{x} + \frac{\overset{1/2}{\cancel{B}}}{x-3} + \frac{\overset{-11/2}{\cancel{C}}}{x+3}$

$-3x^2+18x-18 = A(x-3)(x+3) + Bx(x+3) + Cx(x-3)$

$\underline{x=0}$

$-18 = A\cdot(-3)(3) + B\cdot 0 + C\cdot 0$

$-18 = -9A$

$A = 2$

$\underline{x=3}$ $\quad -3\cdot 9 + 18\cdot 3 - 18 = A\cdot 0 + B\cdot 3\cdot 6 + C\cdot 0$

$9 = 18B$

$\frac{1}{2} = B$

$\underline{x=-3}$ $\quad -3\cdot 9 + 18(-3) - 18 = A\cdot 0 + B\cdot 0 + C\cdot(-6)\cdot(-3)$

$-99 = 18C$

$C = -11/2$

(f) $\int \frac{6x^2-7x+6}{\underbrace{x^3+3x^2}_{x^2(x+3)}}\,dx \quad = \int \frac{-3}{x} + \frac{2}{x^2} + \frac{9}{x+3}\,dx$

$= -3\ln|x| - 2x^{-1} + 9\ln|x+3| + C$

$\frac{6x^2-7x+6}{x^2(x+3)} = \frac{\overset{-3}{\cancel{A}}}{x} + \frac{\overset{2}{\cancel{B}}}{x^2} + \frac{\overset{9}{\cancel{C}}}{x+3}$

$6x^2-7x+6 = Ax(x+3) + B(x+3) + C\cdot x^2$

$\underline{x=0}$ $\quad 6 = A\cdot 0 + B\cdot 3 + C\cdot 0$

$6 = 3B$

$B = 2$

$\underline{x=-3}$ $\quad 6\cdot 9 - 7\cdot(-3) + 6 = A\cdot 0 + B\cdot 0 + C\cdot(-3)^2$

$81 = 9C$

$C = 9$

$\underline{x=1}$ $\quad 6-7+6 = A\cdot 4 + B\cdot 4 + C$

$5 = 4A + 4B + C \quad \boxed{B=2,\ C=9}$

$5 = 4A + 8 + 9$

$-12 = 4A$

$A = -3$

(g) $\displaystyle\int \frac{-x^2+4x+4}{\underbrace{x^3-4x^2+4x}_{x(x^2-4x+4)}}\,dx$

$= x(x-2)^2$

$$\frac{-x^2+4x+4}{x(x-2)^2} = \frac{\cancel{A}^{\,1}}{x} + \frac{\cancel{B}^{\,-2}}{x-2} + \frac{\cancel{C}^{\,4}}{(x-2)^2}$$

$-x^2+4x+4 = A(x-2)^2 + Bx(x-2) + Cx$

$\underline{x=0}$

$4 = A\cdot 4 + B\cdot 0 + C\cdot 0$

$4 = 4A, \quad A = 1$

$\underline{x=2}$ $\quad -4+8+4 = A\cdot 0 + B\cdot 0 + 2C$

$8 = 2C$

$C = 4$

$\underline{x=1}$ $\quad -1+4+4 = A - B + C$

$7 = 1 - B + 4$

$B = -2$

$$= \int \frac{1}{x} - \frac{2}{x-2} + \frac{4}{(x-2)^2}\,dx$$

$$= \ln|x| - 2\ln|x-2| - 4(x-2)^{-1} + C$$

OR

$$= \ln\frac{|x|}{(x-2)^2} - \frac{4}{x-2} + C$$

(h) $\displaystyle\int \frac{x^2-5x-8}{\underbrace{x^3+4x}_{x(x^2+4)}}\,dx$

$\rightarrow$ irreducible

$$\frac{x^2-5x-8}{x(x^2+4)} = \frac{A}{x} + \frac{Bx+C}{x^2+4}$$

$x^2-5x-8 = A(x^2+4) + (Bx+C)x$

$x^2-5x-8 = Ax^2+4A+Bx^2+Cx$

$x^2-5x-8 = (A+B)x^2 + Cx + 4A$

$4A = -8, \quad A = -2$

$C = -5$

$A+B = 1, \quad B = 3$

$$= \int \frac{-2}{x} + \frac{3x-5}{x^2+4}\,dx$$

$$= \int \frac{-2}{x} + \frac{3x}{x^2+4} - \frac{5}{x^2+4}\,dx$$

$$= -2\ln|x| + 3\cdot\frac{1}{2}\ln(x^2+4) - 5\cdot\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + C$$

$$= -2\ln|x| + \frac{3}{2}\ln(x^2+4) - \frac{5}{2}\tan^{-1}\left(\frac{x}{2}\right) + C$$

Remark:

$$\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$$

$$\int \frac{x}{x^2+b}\,dx = \frac{1}{2}\ln|x^2+b| + C$$

(i) $\int \frac{3x^2 - 10x + 15}{(x-1)\underbrace{(x^2+2x+5)}_{\substack{\Delta = 2^2 - 4\cdot 5 \\ = -16 < 0 \\ \text{irreducible}}}}\,dx$

Hint: Complete the square for $x^2 + 2x + 5$

$$\frac{3x^2-10x+15}{(x-1)(x^2+2x+5)} = \frac{\overset{1}{\cancel{A}}}{x-1} + \frac{\overset{2x-10}{\cancel{Bx+C}}}{x^2+2x+5}$$

$$3x^2-10x+15 = A(x^2+2x+5) + (Bx+C)(x-1)$$

$\underline{x=1}$ $\quad 3-10+15 = A\cdot 8 + (B+C)\cdot 0$

$$8 = 8A$$
$$A = 1$$

$$3x^2-10x+15 = (A+B)x^2 + (2A+C-B)x + 5A-C$$

$A+B=3$, $A=1$ so $B=2$

$2A+C-B=-10$, $2+C-2=-10$

$C=-10$

$$\int \frac{3x^2-10x+15}{(x-1)(x^2+2x+15)}\,dx$$

$$= \int \frac{1}{x-1} + \frac{2x-10}{x^2+2x+5}\,dx$$

complete the square

$$= \int \frac{1}{x-1}\,dx + \int \frac{2x-10}{(x+1)^2+4}\,dx$$

$u = x+1$

$du = dx$

$x = u-1$

$$= \ln|x-1| + \int \frac{2(u-1)-10}{u^2+4}\,du$$

$$= \ln|x-1| + \int \frac{2u-12}{u^2+4}\,du$$

$$= \ln|x-1| + \int \frac{2u}{u^2+4} - \frac{12}{x^2+4}\,du$$

$$= \ln|x-1| + 2\cdot\frac{1}{2}\ln|u^2+4| - 12\cdot\frac{1}{2}\tan^{-1}\frac{u}{2} + C$$

$$= \ln|x-1| + \ln[(x+1)^2+4] - 6\tan^{-1}\left(\frac{x+1}{2}\right) + C$$

OR

$$= \ln|x-1|\cdot(x^2+2x+5) - 6\tan^{-1}\left(\frac{x+1}{2}\right) + C$$