Worksheet 7.3

Full Name: ______________________________ Score: ________

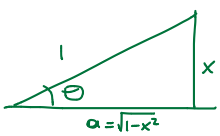
1. Use the method of trigonometric substitution to evaluate each of the following indefinite integrals.

(a) $\int \frac{2x}{\sqrt{1-x^2}}\,dx$ $= \int \frac{2\sin\theta}{\sqrt{1-\sin^2\theta}} \cos\theta \, d\theta$

$x = \sin\theta$
$dx = \cos\theta \, d\theta$

$= \int \frac{2\sin\theta\cos\theta}{\sqrt{\cos^2\theta}}\, d\theta$

$= \int \frac{2\sin\theta\cos\theta}{\cos\theta}\, d\theta$



$\sin\theta = x$

$= \int 2\sin\theta \, d\theta$

$= -2\cos\theta$

$a^2 + x^2 = 1$

$a^2 = 1 - x^2$

$a = \sqrt{1-x^2}$

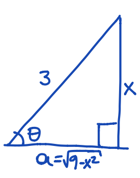
$= -2\sqrt{1-x^2} + C$

$\cos\theta = \sqrt{1-x^2}$

(b) $\int \frac{4x}{9-x^2}\,dx$ $= \int \frac{4\cdot 3\sin\theta}{9-9\sin^2\theta} \cdot 3\cos\theta \, d\theta$

$x = 3\sin\theta$
$dx = 3\cos\theta \, d\theta$

$= \int \frac{36 \sin\theta\cos\theta}{9(1-\sin^2\theta)}\, d\theta$



$= \int 4 \frac{\sin\theta\cos\theta}{\cos^2\theta}\, d\theta$

$= 4\int \frac{\sin\theta}{\cos\theta}\, d\theta$

$u = \cos\theta$
$du = -\sin\theta \, d\theta$

$a^2 + x^2 = 9$
$a = \sqrt{9-x^2}$

$= 4\int -\frac{1}{u}\, du$

$= -4\ln|u| + C$

$\cos\theta = \frac{\sqrt{9-x^2}}{3}$

$= -4\ln|\cos\theta| + C$

$= -4\ln\left(\frac{\sqrt{9-x^2}}{3}\right) + C$

(c) $\displaystyle\int \frac{5x}{4+25x^2}\,dx$ $\displaystyle = \int \frac{5\cdot\frac{2}{5}\tan\theta}{4+25\left(\frac{2}{5}\tan\theta\right)^2}\cdot\frac{2}{5}\sec^2\theta\,d\theta$

$x = \frac{2}{5}\tan\theta$

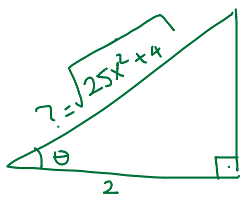
$dx = \frac{2}{5}\sec^2\theta\,d\theta$

$$= \int \frac{2\tan\theta\cdot\frac{2}{5}\sec^2\theta}{4+\cancel{25}\cdot\frac{4}{\cancel{25}}\tan^2\theta}\,d\theta$$

$$= \int \frac{\frac{4}{5}\tan\theta\sec^2\theta}{4(1+\tan^2\theta)}\,d\theta$$

$$= \int \frac{\frac{\cancel{4}}{5}\tan\theta\sec^2\theta}{\cancel{4}\sec^2\theta}\,d\theta$$

$$= \frac{1}{5}\int \tan\theta\,d\theta$$



$x = \frac{2}{5}\tan\theta$

$\frac{5x}{2} = \tan\theta = \frac{\text{opp}}{\text{adj}}$

$$= \frac{1}{5}\int \frac{\sin\theta}{\cos\theta}\,d\theta$$

$u = \cos\theta$
$du = -\sin\theta\,d\theta$

$$= -\frac{1}{5}\int \frac{1}{u}\,du$$

$$= -\frac{1}{5}\ln|u| + C$$

$$= -\frac{1}{5}\ln|\cos\theta| + C$$

$\cos\theta = \frac{2}{\sqrt{25x^2+4}}$

$$= -\frac{1}{5}\ln\left(\frac{2}{\sqrt{25x^2+4}}\right) + C$$

(d) $\displaystyle\int \frac{x}{16x^2-9}\,dx$ $\displaystyle = \int \frac{\frac{3}{4}\sec\theta}{\cancel{16}\cdot\frac{9}{\cancel{16}}\sec^2\theta-9}\cdot\frac{3}{4}\sec\theta\tan\theta\,d\theta$

$x = \frac{3}{4}\sec\theta$

$dx = \frac{3}{4}\sec\theta\tan\theta\,d\theta$

$$= \int \frac{\frac{9}{16}\cdot\sec^2\theta\tan\theta}{9(\sec^2\theta-1)}\,d\theta$$

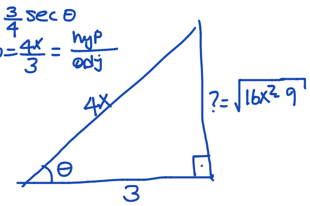
$$= \frac{1}{16}\int \frac{\sec^2\theta\tan\theta}{\tan^2\theta}\,d\theta$$

$$= \frac{1}{16}\int \frac{\sec^2\theta}{\tan\theta}\,d\theta$$

$u = \tan\theta$
$du = \sec^2\theta\,d\theta$

$x = \frac{3}{4}\sec\theta$

$\sec\theta = \frac{4x}{3} = \frac{\text{hyp}}{\text{adj}}$



$$= \frac{1}{16}\int \frac{1}{u}\,du$$

$$= \frac{1}{16}\ln|u| + C$$

$$= \frac{1}{16}\ln|\tan\theta| + C$$

$\tan\theta = \frac{\sqrt{16x^2-9}}{3}$

$$= \frac{1}{16}\ln\frac{\sqrt{16x^2-9}}{3} + C$$

(e) $\displaystyle\int \frac{1}{\sqrt{16+x^2}}\,dx = \int \frac{1}{\sqrt{16+16\tan^2\theta}} \cdot 4\sec^2\theta\, d\theta$

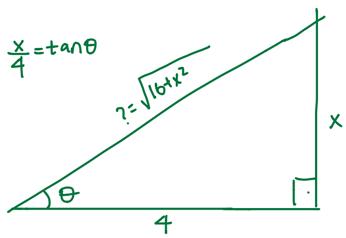
$x = 4\tan\theta$

$dx = 4\sec^2\theta$

$$= \int \frac{4\sec^2\theta}{\sqrt{16(1+\tan^2\theta)}}\, d\theta$$

$$= \int \frac{4\sec^2\theta}{4\sec\theta}\, d\theta$$

$\frac{x}{4} = \tan\theta$



$\sec\theta = \frac{\sqrt{16+x^2}}{4}$

$$= \int \sec\theta\, d\theta$$

$$= \ln|\sec\theta + \tan\theta| + C$$

$$= \ln\left|\frac{\sqrt{16+x^2}}{4} + \frac{x}{4}\right| + C$$

(f) $\int \frac{x^5}{\sqrt{x^2+4}}\,dx = \int \frac{32\tan^5\theta}{\sqrt{4\tan^2\theta+4}} \cdot 2\sec^2\theta\, d\theta$

$x = 2\tan\theta$

$dx = 2\sec^2\theta\, d\theta$

$$= \int \frac{64\tan^5\theta\sec^2\theta}{\sqrt{4(\tan^2\theta+1)}}\, d\theta$$

$$= \int \frac{64\tan^5\theta\sec^2\theta}{2\sec\theta}\, d\theta$$

$$= \int 32\tan^5\theta\cdot\sec\theta\, d\theta$$

(g) $\int x^2\sqrt{9-x^2}\,dx = \int 9\sin^2\theta\sqrt{9-9\sin^2\theta}\;3\cos\theta\,d\theta$

$x=3\sin\theta$
$dx=3\cos\theta\,d\theta$

$$= \int 27\sin^2\theta\cos\theta\sqrt{9\cos^2\theta}\,d\theta$$

$$= \int 27\sin^2\theta\cos\theta\;3\cos\theta\,d\theta$$

$$= \int 81\sin^2\theta\cos^2\theta\,d\theta$$

$$= \int 81\cdot\frac{1}{2}(1-\cos 2\theta)\cdot\frac{1}{2}(1+\cos 2\theta)\,d\theta$$

$$= \int \frac{81}{4}(1-\cos^2(2\theta))\,d\theta$$

$$= \frac{81}{4}\int 1-\frac{1}{2}(1+\cos 4\theta)\,d\theta$$

$$= \frac{81}{4}\int 1-\frac{1}{2}-\frac{1}{2}\cos 4\theta\,d\theta$$

$$= \frac{81}{4}\int \frac{1}{2}-\frac{1}{2}\cos 4\theta\,d\theta$$

$$= \frac{81}{4}\left(\frac{1}{2}\theta-\frac{1}{2}\cdot\frac{1}{4}\sin 4\theta\right)+C$$

$$= \frac{81}{8}\theta-\frac{81}{32}\sin 4\theta+C$$

$$= \frac{81}{8}\theta-\frac{81}{32}\cdot 2\sin 2\theta\cos 2\theta+C$$

$$= \frac{81}{8}\theta-\frac{81}{32}\cdot 2\cdot 2\sin\theta\cdot\cos\theta(\cos^2\theta-\sin^2\theta)+C$$

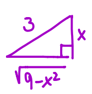
$$= \frac{81}{8}\theta-\frac{81}{8}\sin\theta\cos\theta(\cos^2\theta-\sin^2\theta)+C$$

$$= \frac{81}{8}\sin^{-1}\left(\frac{x}{3}\right)-\frac{81}{8}\cdot\frac{x}{3}\cdot\frac{\sqrt{9-x^2}}{3}\left(\frac{9-x^2}{9}-\frac{x^2}{9}\right)+C$$

$$= \frac{81}{8}\sin^{-1}\left(\frac{x}{3}\right)-\frac{9}{8}x\sqrt{9-x^2}\cdot\frac{9-2x^2}{9}+C$$

$$= \frac{81}{8}\sin^{-1}\left(\frac{x}{3}\right)-\frac{x\cdot\sqrt{9-x^2}\cdot(9-2x^2)}{8}+C$$

$x=3\sin\theta$
$\frac{x}{3}=\sin\theta$
$\theta=\sin^{-1}\left(\frac{x}{3}\right)$



$\frac{x}{3}=\sin\theta$

$\cos\theta=\frac{\sqrt{9-x^2}}{3}$

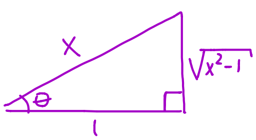
(h) $\int \frac{\sqrt{x^2-1}}{x}\,dx = \int \frac{\sqrt{\sec^2\theta - 1}}{\sec\theta} \cdot \sec\theta\tan\theta\, d\theta$

$x = \sec\theta$

$dx = \sec\theta\tan\theta\, d\theta$

$$= \int \frac{\tan\theta}{\sec\theta} \sec\theta\tan\theta\, d\theta$$

$$= \int \tan^2\theta\, d\theta$$



$x = \sec\theta = \frac{\text{hyp}}{\text{adj}}$

$$= \int (\sec^2\theta - 1)\, d\theta$$

$$= \tan\theta - \theta + C$$

$\tan\theta = \frac{\text{opp}}{\text{adj}} = \frac{\sqrt{x^2-1}}{1} = \sqrt{x^2-1}$

$$= \sqrt{x^2-1} - \sec^{-1}(x) + C$$