Worksheet 6.5

Solutions

1. Find the average value of the function on the given interval.

(a) $f(x) = x^2 + x + 1$ on $[1, 3]$

Solution:

$$\overline{f} = \frac{1}{3-1}\int_1^3 x^2 + x + 1 \, dx = \frac{1}{2}\left[\frac{x^3}{3} + \frac{x^2}{2} + x\right]_1^3 = \frac{1}{2}\left[\left(9 + \frac{9}{2} + 3\right) - \left(\frac{1}{3} + \frac{1}{2} + 1\right)\right] = \frac{22}{3}.$$

(b) $f(x) = \dfrac{1}{x}$ on $[1, e^2]$

Solution:

$$\overline{f} = \frac{1}{e^2 - 1}\int_1^{e^2} \frac{1}{x} \, dx = \frac{1}{e^2-1}\Big[\ln|x|\Big]_1^{e^2} = \frac{\ln e^2 - \ln 1}{e^2 - 1} = \frac{2}{e^2-1}.$$

(c) $f(x) = \dfrac{2x}{(1+x^2)^2}$ on $[0, 2]$

Solution: Let $u = 1 + x^2$, then $du = 2x \, dx$.

$$\overline{f} = \frac{1}{2-0}\int_0^2 \frac{2x}{(1+x^2)^2} \, dx = \frac{1}{2}\int_{u(0)}^{u(2)} \frac{1}{u^2} \, du = \frac{1}{2}\left[-\frac{1}{u}\right]_{u(0)}^{u(2)} = \frac{1}{2}\left[-\frac{1}{1+x^2}\right]_0^2 = \frac{1}{2}\left(-\frac{1}{5} + 1\right) = \frac{2}{5}.$$

(d) $f(t) = t\sin(t^2)$ on $[0, 10]$

Solution: Let $u = t^2$, then $du = 2t \, dt$.

$$\overline{f} = \frac{1}{10-0}\int_0^{10} t\sin(t^2) \, dt = \frac{1}{20}\int_{u(0)}^{u(10)} \sin u \, du = \frac{1}{20}\Big[-\cos u\Big]_{u(0)}^{u(10)} = \left[\frac{-\cos(t^2)}{20}\right]_0^{10} = \frac{1 - \cos 100}{20}.$$

2. The temperature in °C in a city t hours after 09:00 is modelled by the function

$$T(t) = 10 + 8\sin\left(\frac{\pi t}{12}\right).$$

Find the average temperature in that city in between 09:00 and 21:00.

Solution: The average temperature is

$$T_{\text{ave}} = \frac{1}{12-0}\int_0^{12} 10 + 8\sin\left(\frac{\pi t}{12}\right) \, dt = \frac{1}{12}\left[10t - \frac{12}{\pi}\cdot 8\cos\left(\frac{\pi t}{12}\right)\right]_0^{12} = \left[\frac{10t}{12} - \frac{8}{\pi}\cos\left(\frac{\pi t}{12}\right)\right]_0^{12},$$

whence

$$T_{\text{ave}} = \left[\frac{10(12)}{12} - \frac{8}{\pi}\cos\left(\frac{\pi(12)}{12}\right)\right] - \left[\frac{10(0)}{12} - \frac{8}{\pi}\cos\left(\frac{\pi(0)}{12}\right)\right].$$

Ergo,

$$T_{\text{ave}} = 10 - \frac{8}{\pi}\cos(\pi) - 0 + \frac{8}{\pi}\cos(0) = 10 + \frac{8}{\pi} + \frac{8}{\pi} = 10 + \frac{16}{\pi} \text{ °C}.$$

3. A patient being treated for pulmonary fibrosis is tested with a spirometer to measure lung capacity. The data show the volume of air in the patient's lung during both the inhalation and exhalation cycles is given by

$$V(t) = 1 - \cos\left(\frac{2\pi t}{5}\right) \text{ pints}$$

over a period from $t = 0$ seconds till $t = 5$ seconds. Find the average volume of air in his lungs during this period. At what time(s) does this volume occur?

Solution: The average volume is

$$V_{\text{ave}} = \frac{1}{5-0}\int_0^5 1 - \cos\left(\frac{2\pi t}{5}\right)\, \mathrm{d}t = \frac{1}{5}\left[t - \frac{5}{2\pi}\sin\left(\frac{2\pi t}{5}\right)\right]_0^5 = \left[\frac{t}{5} - \frac{1}{2\pi}\sin\left(\frac{2\pi t}{5}\right)\right]_0^5,$$

whence

$$V_{\text{ave}} = \left[\frac{5}{5} - \frac{1}{2\pi}\sin\left(\frac{2\pi(5)}{5}\right)\right] - \left[\frac{0}{5} - \frac{1}{2\pi}\sin\left(\frac{2\pi(0)}{5}\right)\right].$$

Ergo,

$$V_{\text{ave}} = 1 - \frac{1}{2\pi}\sin(2\pi) - 0 + \frac{1}{2\pi}\sin(0) = 1 \text{ pint}.$$

This volume occurs when $V(t) = 1$, i.e.

$$1 - \cos\left(\frac{2\pi t}{5}\right) = 1.$$

Solving for the cosine term, we have

$$\cos\left(\frac{2\pi t}{5}\right) = 0.$$

The general solution to this trigonometric equation includes all odd integer multiples of $\pi/2$:

$$\frac{2\pi t}{5} = (2k+1)\frac{\pi}{2}, \quad k \in \mathbb{Z}^+.$$

Thus

$$t = (2k+1)\frac{\pi}{2} \cdot \frac{5}{2\pi} = \frac{5(2k+1)}{4} \text{ seconds}.$$

Out of the infinitely many t values

$$t = \frac{5}{4}, \frac{15}{4}, \frac{25}{4}, \ldots,$$

only the first two lie within the interval $[0, 5]$. Therefore, the volume of $V(t) = 1$ pint occurs at

$$t = \frac{5}{4}, \frac{15}{4} \text{ seconds}.$$