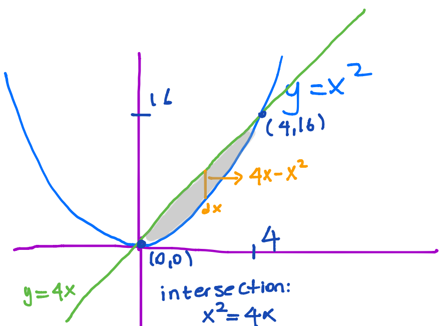
Worksheet 6.1

Full Name: ______________________________ Score: ________

1. Sketch the region enclosed by the graphs of the given equations. Then, use a definite integral to find the exact value of the area of the region.

   (a) $y = x^2, \quad y = 4x$



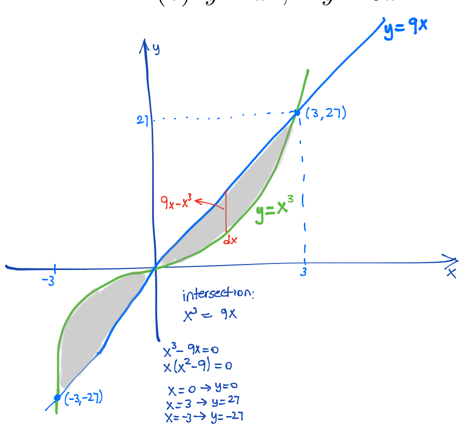
intersection:
$x^2 = 4x$
$x^2 - 4x = 0$
$x(x-4) = 0$
$x=0 \to y=0$
$x=4 \to y=16$

$$\text{Area} = \int_0^4 4x - x^2\,dx = 2x^2 - \tfrac{1}{3}x^3 \Big|_0^4$$
$$= 2\cdot 4^2 - \tfrac{1}{3}4^3 - 0$$
$$= 32 - \tfrac{64}{3} = \tfrac{32}{3}$$

$y=x^2$, $x=\sqrt{y}$, $y=4x$, $x=\frac{y}{4}$

OR
$$\text{Area} = \int_0^{16} \sqrt{y} - \frac{y}{4}\,dy$$
$$= \tfrac{2}{3}y^{3/2} - \tfrac{1}{8}y^2 \Big|_0^{16}$$
$$= \left(\tfrac{2}{3}16^{3/2} - \tfrac{1}{8}\cdot 16^2\right) - 0$$
$$= \frac{128}{3} - 2\cdot 16 = \frac{32}{3}$$

   (b) $y = x^3, \quad y = 9x$



intersection:
$x^3 = 9x$
$x^3 - 9x = 0$
$x(x^2-9) = 0$
$x=0 \to y=0$
$x=3 \to y=27$
$x=-3 \to y=-27$

use dx:
The region in the lower part is the same as the upper one. So,

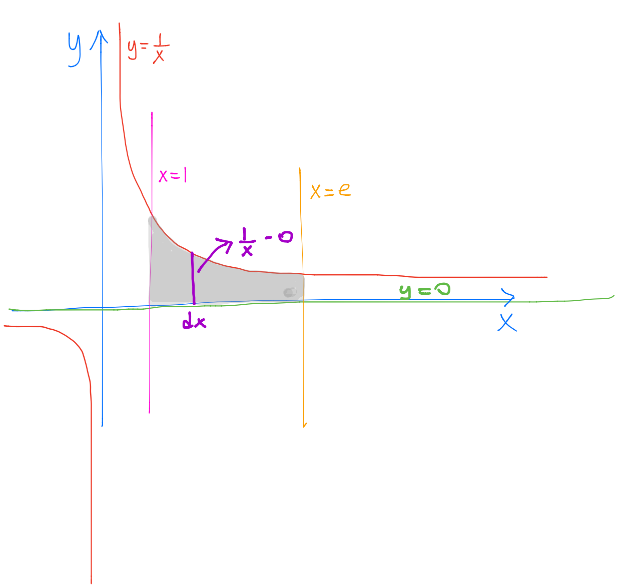
$$A = 2\int_0^3 9x - x^3\,dx$$
or $A = \int_{-3}^0 x^3 - 9x\,dx + \int_0^3 9x - x^3\,dx$
$$= 2\left(\tfrac{9}{2}x^2 - \tfrac{1}{4}x^4\right)\Big|_0^3$$
$$= 9x^2 - \tfrac{1}{2}x^4\Big|_0^3$$
$$= \left(9\cdot 3^2 - \tfrac{1}{2}\cdot 3^4\right) - 0$$
$$= 81 - \tfrac{81}{2}$$
$$= \tfrac{81}{2}$$

OR,
use dy:

$y=x^3$, $x=\sqrt[3]{y}$, $y=9x$, $x=\frac{y}{9}$

$$A = 2\int_0^{27} \sqrt[3]{y} - \frac{y}{9}\,dy \quad \left(\text{or } A = \int_{-27}^0 \frac{y}{9} - \sqrt[3]{y}\,dy + \int_0^{27} \sqrt[3]{y} - \frac{y}{9}\,dy\right)$$
$$= \tfrac{3}{2}\cdot y^{4/3} - \tfrac{1}{9}\cdot y^2\Big|_0^{27}$$
$$= \left(\tfrac{3}{2}(27)^{4/3} - \tfrac{1}{9}(27)^2\right) - 0$$
$$= \left(\tfrac{3}{2}\cdot 81 - 3\cdot 27\right)$$
$$= \frac{243-162}{2} = \frac{81}{2}$$

(c) $y = \dfrac{1}{x}, \quad y = 0, \quad x = 1, \quad x = e$



$$\text{Area} = \int_1^e \frac{1}{x}\,dx$$

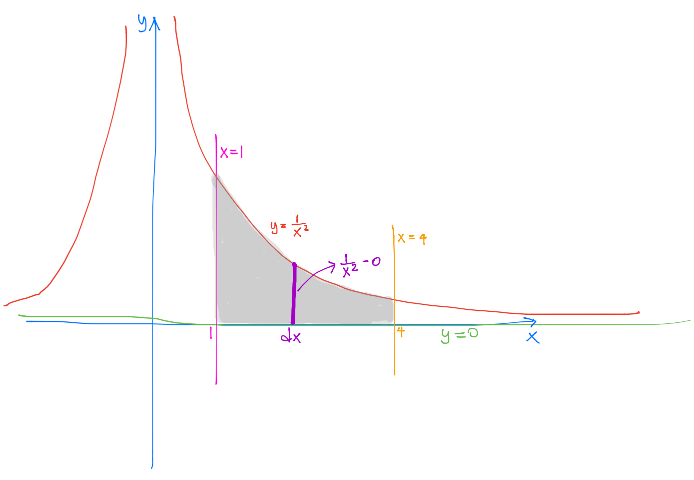
$$= \ln|x| \Big|_1^e$$

$$= \ln e - \ln 1$$

$$= 1 - 0$$

$$= 1$$

(d) $y = \dfrac{1}{x^2}, \quad y = 0, \quad x = 1, \quad x = 4$



$$\text{Area} = \int_1^4 \frac{1}{x^2}\,dx$$

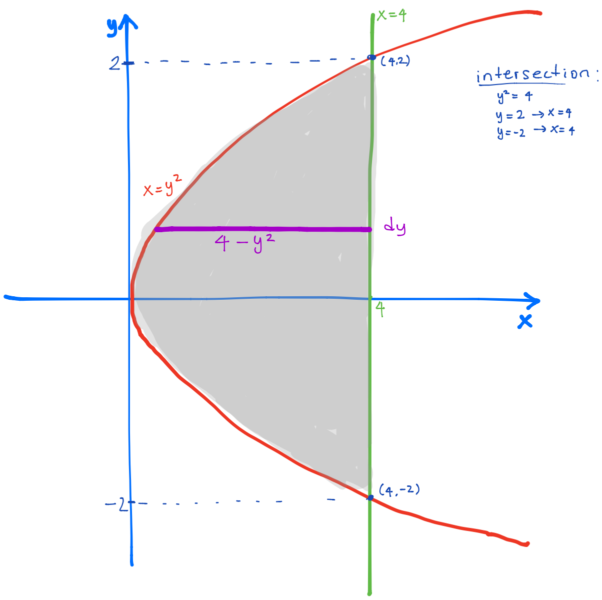
$$= -\frac{1}{x} \Big|_1^4$$

$$= \left(-\frac{1}{4}\right) - \left(-\frac{1}{1}\right)$$

$$= -\frac{1}{4} + 1$$

$$= \frac{3}{4}$$

(e) $x = y^2, \quad x = 4$



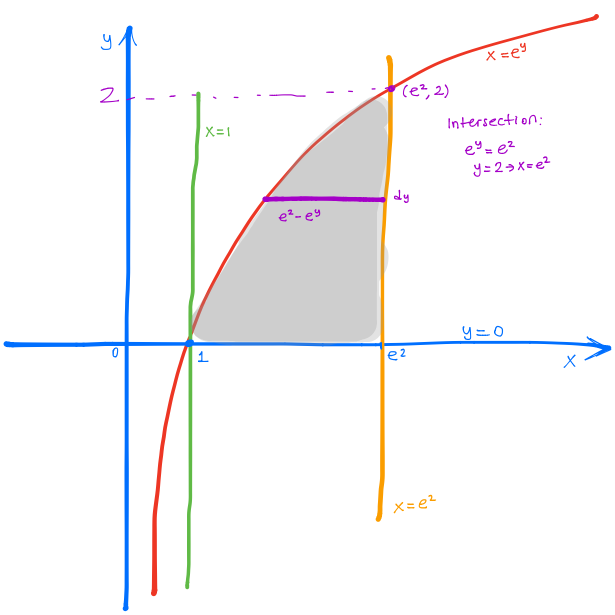
intersection:
$y^2 = 4$
$y = 2 \to x = 4$
$y = -2 \to x = 4$

$$\text{Area} = \int_{-2}^{2} 4 - y^2 \, dy$$
$$= 4y - \tfrac{1}{3}y^3 \Big|_{-2}^{2}$$
$$= \left(8 - \tfrac{1}{3}\cdot 8\right) - \left(-8 + \tfrac{1}{3}8\right)$$
$$= 8 - \tfrac{8}{3} + 8 - \tfrac{8}{3}$$
$$= 16 - \tfrac{16}{3}$$
$$= \tfrac{32}{3}$$

we could also use the symmetry:

$$\text{Area} = 2\int_{0}^{2} 4 - y^2 dy$$
$$= 2\left(4y - \tfrac{1}{3}y^3\right)\Big|_0^2$$
$$= 8y - \tfrac{2}{3}y^3 \Big|_0^2$$
$$= \left(16 - \tfrac{2}{3}\cdot 8\right) - 0$$
$$= 16 - \tfrac{16}{3}$$
$$= \tfrac{32}{3}$$

(f) $x = e^y, \quad y = 0, \quad x = 1, \quad x = e^2$



Intersection:
$e^y = e^2$
$y = 2 \Rightarrow x = e^2$

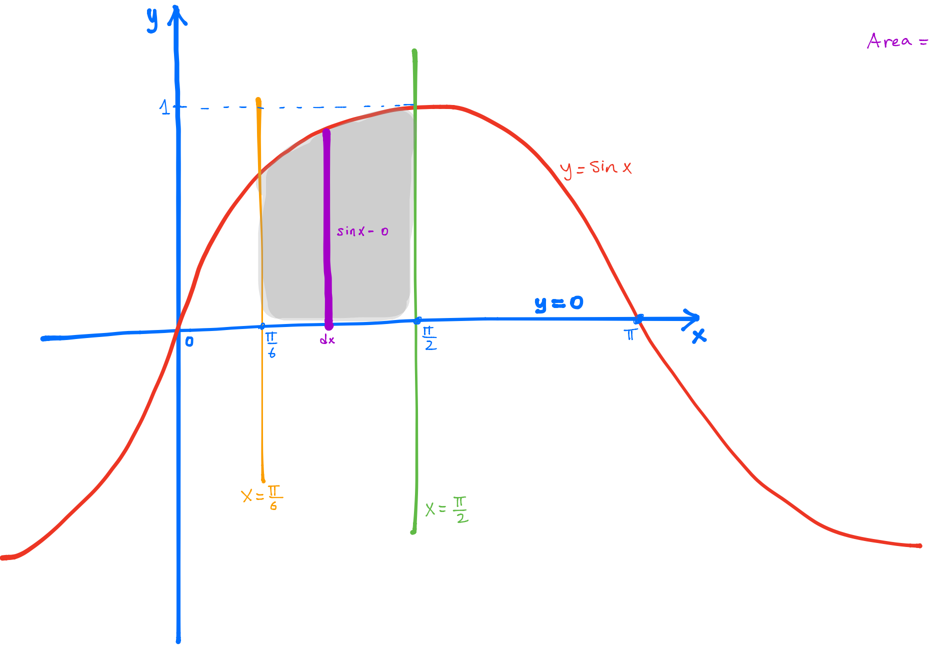
$$\text{Area} = \int_0^2 e^2 - e^y \, dy$$
$$= e^2 y - e^y \Big|_0^2$$
$$= \left(e^2 2 - e^2\right) - \left(e^2 0 - e^0\right)$$
$$= 2e^2 - e^2 - 0 + 1$$
$$= e^2 + 1$$

Integral with respect to x also works:

$x = e^y$
$\ln x = \ln e^y$
$\ln x = y$
$y = \ln x$

$\text{Area} = \int_1^{e^2} \ln x \, dx \longrightarrow$ However, we need integration by parts (Section 7.1) to evaluate this integral.

(g) $y = \sin x, \quad y = 0, \quad x = \frac{\pi}{6}, \quad x = \frac{\pi}{2}$



$$\text{Area} = \int_{\pi/6}^{\pi/2} \sin x \; dx$$

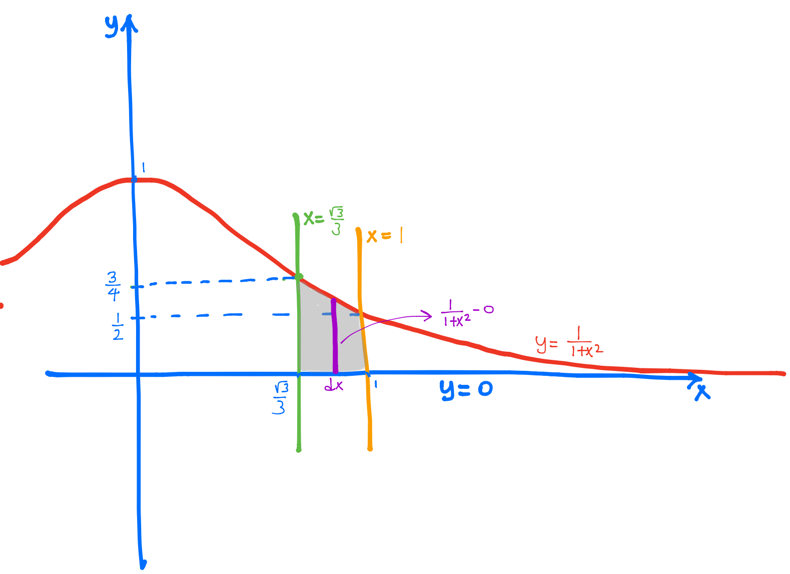
$$= -\cos x \Big|_{\pi/6}^{\pi/2}$$

$$= \left(-\cos\frac{\pi}{2}\right) - \left(-\cos\frac{\pi}{6}\right)$$

$$= 0 - \left(-\frac{\sqrt{3}}{2}\right)$$

$$= \frac{\sqrt{3}}{2}$$

(h) $y = \frac{1}{1+x^2}, \quad y = 0, \quad x = \frac{\sqrt{3}}{3}, \quad x = 1$



$$\text{Area} = \int_{\sqrt{3}/3}^{1} \frac{1}{1+x^2} \, dx$$

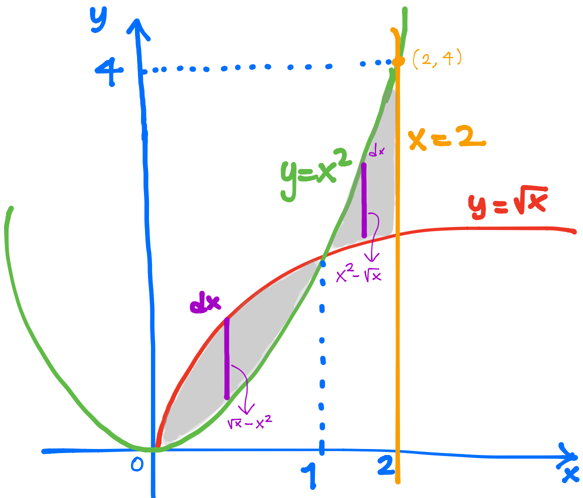
$$= \tan^{-1} x \Big|_{\sqrt{3}/3}^{1}$$

$$= \tan^{-1}(1) - \tan^{-1}\left(\frac{\sqrt{3}}{3}\right)$$

$$= \frac{\pi}{4} - \frac{\pi}{6}$$

$$= \frac{\pi}{12}$$

(i) $y = \sqrt{x}, \quad y = x^2, \quad x = 2$



Intersection:

$\sqrt{x} = x^2$

$x = x^4$

$0 = x^4 - x$

$0 = x(x^3 - 1)$

$0 = x\cdot(x-1)(x^2+x+1)$

$x = 0$, $x - 1 = 0$, $x^2 + x + 1 = 0$ ← No sol

$x = 0$, $x = 1$

Since $y=\sqrt{x}$ is the bigger function on $[0,1]$ & $y=x^2$ is the bigger function on $[1,2]$, we split the interval $[0,2]$ into two intervals.

$$\text{Area} = \int_0^1 \sqrt{x} - x^2 \, dx + \int_1^2 x^2 - \sqrt{x} \, dx$$

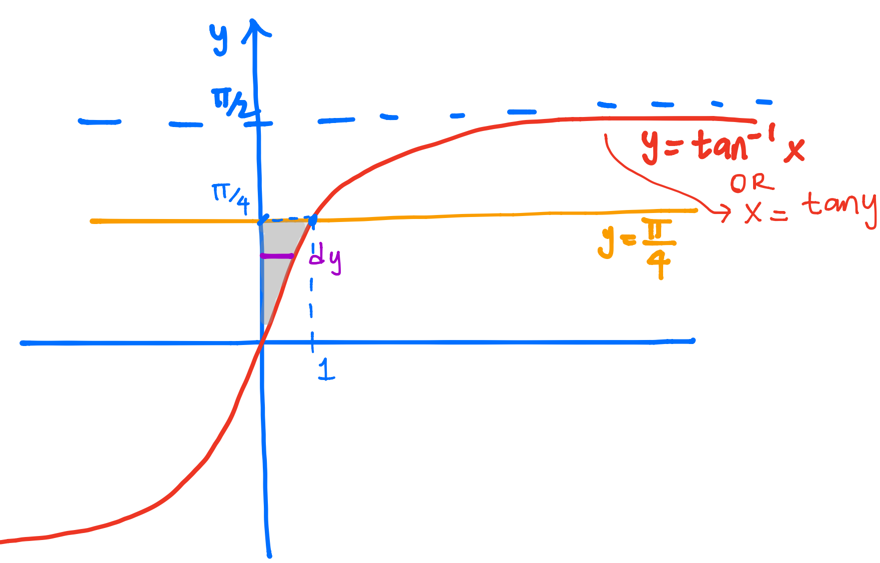
$$= \left(\frac{2}{3}x^{3/2} - \frac{1}{3}x^3\right)\Big|_0^1 + \left(\frac{1}{3}x^3 - \frac{2}{3}x^{3/2}\right)\Big|_1^2$$

$$= \left(\frac{2}{3} - \frac{1}{3}\right) - 0 + \left(\frac{1}{3}\cdot 2^3 - \frac{2}{3}\cdot 2^{3/2}\right) - \left(\frac{1}{3} - \frac{2}{3}\right)$$

$$= \frac{2}{3} + \frac{8}{3} + \frac{4\sqrt{2}}{3}$$

$$= \frac{10}{3} + \frac{4\sqrt{2}}{3}$$

(j) $y = \arctan x, \quad y = \frac{\pi}{4}, \quad y\text{-axis}$



$$\text{Area} = \int_0^{\pi/4} \tan y \, dy$$

$$= \int_0^{\pi/4} \frac{\sin y}{\cos y} \, dy$$

$u = \cos y$; $y = \pi/4 \to u = \cos \pi/4 = \frac{\sqrt{2}}{2}$; $y = 0 \to u = \cos 0 = 1$

$du = -\sin y \, dy$

$$\text{Area} = \int_1^{\sqrt{2}/2} -\frac{1}{u} \, du$$

$$= -\ln|u| \Big|_1^{\sqrt{2}/2}$$

$$= -\ln\left(\frac{\sqrt{2}}{2}\right) - (-\ln 1)$$

$$= -\ln\left(\frac{\sqrt{2}}{2}\right) + 0$$

$$\approx 0.34657$$

We could also write:

$$\text{Area} = \int_0^1 \left(\frac{\pi}{4} - \tan^{-1} x\right) dx$$

But, we don't know how to find the integral of $\tan^{-1} x$ yet.