Worksheet 11.3

Full Name: ______________________________ Score: ________

1. Determine if each of the following **p-series** is convergent or divergent.

(a) $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges $p = 3 > 1$

(b) $\sum_{n=1}^{\infty} \frac{1}{\sqrt[5]{n^3}}$ diverges $p = 3/5 < 1$

(c) $\sum_{n=1}^{\infty} \frac{1}{n^{\sqrt{2}}}$ converges $p = \sqrt{2} > 1$

(d) $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n^4}}$ converges $p = 4/3 > 1$

(e) $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \ldots$

$= \sum \frac{1}{n^2}$ converges $p = 2$

(f) $1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \frac{1}{\sqrt{5}} + \ldots$

$= \sum \frac{1}{\sqrt{n}}$ diverges $p = \frac{1}{2}$.

2. Use the **integral test** to determine whether the series is convergent or divergent.

(a) $\sum_{n=1}^{\infty} \frac{1}{1+n^2}$ Let $f(x) = \frac{1}{1+x^2}$ then $f(x) > 0$ for all $x > 1$

and f is decreasing.

$$\int_1^\infty \frac{1}{1+x^2}dx = \lim_{t\to\infty} \int_1^t \frac{1}{1+x^2}dx$$

$$= \lim_{t\to\infty} \left(\arctan(x) \Big|_1^t\right)$$

$$= \lim_{t\to\infty} \arctan(t) - \arctan(1)$$

$$= \pi/2 - \pi/4 = \pi/4 \text{ converges}$$

Hence $\sum \frac{1}{1+n^2}$ converges.

(b) $\sum_{n=1}^{\infty} \frac{\sqrt[3]{n}+2}{n^2}$ Let $f(x) = \frac{\sqrt[3]{x}+2}{x^2} = \frac{\sqrt[3]{x}}{x^2} + \frac{2}{x^2}$

$= \frac{1}{x^{5/3}} + \frac{2}{x^2}$

$f(x) > 0$ for all x

and f is decreasing

$$\frac{x^{-5/3+1}}{-5/3+1} = \frac{-2}{3x^{2/3}}$$

$$\int_1^{\infty}\left(\frac{1}{x^{5/3}} + \frac{2}{x^2}\right)dx = \lim_{t\to\infty}\int_1^t\left(\frac{1}{x^{5/3}} + \frac{2}{x^2}\right)dx$$

$$= \lim_{t\to\infty}\left(-\frac{2}{3x^{2/3}} - \frac{2}{x}\right)\Big|_1^t = \lim\left(\frac{-2}{3x^{2/3}} - \frac{2}{x}\right) - \left(\frac{-2}{3} - 2\right)$$

$$= \tfrac{2}{3} + 2 - 0 = \tfrac{2}{3} + 2$$

Converges

$\sum_{n=1}^{\infty} \frac{\sqrt[3]{n}+2}{n^2}$ converges.

(c) $\sum_{n=1}^{\infty} \frac{n}{n^2+4}$

Let $f(x) = \frac{x}{x^2+4}$, $f(x) > 0$ and f is decreasing

$$\int_1^{\infty}\frac{x}{x^2+4}dx = \lim_{t\to\infty}\int_1^t \frac{x}{x^2+4}dx$$

Let $u = x^2+4 \Rightarrow du = 2x\,dx$

$$\int_1^t \frac{x}{x^2+4}dx = \frac{1}{2}\int_5^{t^2+4}\frac{1}{u}du = \frac{1}{2}\ln|u|\Big|_5^{t^2+4}$$

$$= \tfrac{1}{2}\ln|t^2+4| - \tfrac{1}{2}\ln|5|$$

$$\int_1^{\infty}\frac{x}{x^2+4}dx = \lim_{t\to\infty}\left(\tfrac{1}{2}\ln(t^2+4) - \tfrac{1}{2}\ln(5)\right) = \infty \text{ div}$$

$\sum_{n=1}^{\infty}\frac{n}{n^2+4}$ is divergent

(d) $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ $f(x) = \frac{\ln(x)}{x}$ for $x \geq 3$

Then $f(x) > 0$ and f is decreasing on $(3, \infty)$

$$\int_3^{\infty} \frac{\ln(x)}{x} dx = \lim_{t \to \infty} \int_3^t \frac{\ln(x)}{x} dx$$

But $\int \frac{\ln x}{x} dx = \int u\, du = \frac{u^2}{2} + c = \frac{(\ln(x))^2}{2} + c$

(letting $u = \ln x$)

So $\int_3^{\infty} \frac{\ln(x)}{x} dx = \lim_{t \to \infty} \left(\frac{\ln(t)^2}{2} - \frac{\ln(3)}{2} \right) = \infty$

$\sum_{n=3}^{\infty} \frac{\ln(n)}{n}$ is divergent. Therefore $\sum_{n=1}^{\infty} \frac{\ln(n)}{n}$ is divergent too.

(e) $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$

letting $u = \ln(x)$

$$\int_2^t \frac{1}{x \ln(x)} dx = \ln(\ln(x)) \Big|_2^t = \ln(\ln(t)) - \ln(\ln(2))$$

Therefore $\int_2^{\infty} \frac{1}{x \ln x} dx = \lim_{t \to \infty} \left(\ln(\ln(t)) - \ln(\ln(2)) \right)$

$= \infty$

Therenefore the integral $\int_2^{\infty} \frac{1}{x \ln x} dx$ diverges

the series $\sum \frac{1}{n \ln n}$ diverges too.

(f) $\sum_{n=1}^{\infty} n^2 e^{-n^3}$ $\quad f(x) = x^2 e^{-x^3} = \dfrac{x^2}{e^{x^3}}$

$f(x) > 0$ and f is decreasing.

$$\int x^2 e^{-x^3} dx = \int x^2 e^u \frac{du}{-3x^2} = -\frac{1}{3}\int e^u du = -\frac{1}{3}e^u + c$$
$$= -\frac{1}{3}e^{-x^3} + c$$

$u = -x^3 \Rightarrow du = -3x^2 dx$

$$\int_1^t x^2 e^{-x^3} dx = -\frac{1}{3}e^{-t^3} + \frac{1}{3}e^{-1}$$

$$\int_1^{\infty} x^2 e^{-x^3} dx = \lim_{t\to\infty}\left(\frac{1}{3e} - \frac{1}{e^{t^3}}\right) = \frac{1}{3e}$$

converges

Therefore $\sum_{n=1}^{\infty} n^2 e^{-n^3}$ converges too.

(g) $\sum_{n=1}^{\infty} \dfrac{e^{1/n}}{n^2}$

$$f(x) = \frac{e^{1/x}}{x^2},$$

$$\int \frac{e^{1/x}}{x^2} = -\int e^u du \quad \left(\text{with } u = \frac{1}{x}\right)$$

$$= -e^u + c = -e^{1/x} + c$$

$$\int_1^t \frac{e^{1/x}}{x^2} dx = e - e^{1/t}$$

$$\int_1^{\infty} \frac{e^{1/x}}{x^2} dx = \lim_{t\to\infty}(e - e^{1/t}) = e - 1 \quad \text{converges}$$

Therefore $\sum \dfrac{e^{1/n}}{n^2}$ converges too.