Worksheet 11.1 and 11.2

Full Name: ______________________________ Score: ________

1. Determine if the sequence converges or diverges. If it converges, determine its limit.

(a) $a_n = \dfrac{5n^2+2n+20}{3n^5-3n^2-5}$ converges to 0.

(b) $a_n = \dfrac{9n^4-3n}{-27n^4+100n^2+1000}$ converges to $-1/3$

(c) $a_n = \dfrac{2n^3+1}{1500n^2+100n+3000}$ diverges

$\lim_{n\to\infty} a_n = \infty$

(d) $a_n = 2^n$

$\lim_{n\to\infty} 2^n = \infty$. diverges

(e) $a_n = \left(\dfrac{1}{3}\right)^n$

$\lim_{n\to\infty} a_n = 0$, Converges to 0.

(f) $a_n = 4^{-n}$ $= (1/4)^n$ converges to 0

(g) $a_n = 2^{\frac{n}{n+1}}$ converges to $2^1 = 2$

(h) $a_n = \left(\dfrac{n}{3n+1}\right)^n$ note: $\frac{n}{3n+1} \to 1/3$

so $\left(\frac{n}{3n+1}\right)^n \approx (1/3)^n \to 0$ converges to zero.

(i) $a_n = 3^{\frac{n}{n^2+1}}$ converges to $3^0 = 1$, since

$\lim_{n\to\infty}\left(\frac{n}{n^2+1}\right) = 0$.

(j) $\left\{2, -\dfrac{4}{3}, \dfrac{8}{9}, -\dfrac{18}{27}, \dfrac{32}{81}, \dots\right\}$ $a_n = 2(-2/3)^n$, $n = 0, 1, \dots$

$\lim_{n\to\infty} a_n = 0$.

(k) $\{-4, 12, -36, 108, -324, \dots\}$ diverges.

2. Determine if the series converges or diverges. If it converges, determine its sum.

(a) $\sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n = \frac{1}{1-2/3}$

$= \frac{1}{1/3}$

$= 3$

(b) $\sum_{n=2}^{\infty} \left(\frac{1}{2}\right)^n = \frac{1/4}{1-1/2} = \frac{1/4}{1/2} = 1/2.$

(First term is $\frac{1}{4}$)

(c) $\sum_{n=0}^{\infty} 5\left(-\frac{3}{4}\right)^n = \frac{5}{1+3/4} = \frac{5}{7/4} = \frac{20}{7}$

(d) $\sum_{n=0}^{\infty} \left(\frac{5}{4}\right)^n$ $r = 5/4 > 1$

So the series diverges

(e) $2 - \frac{1}{2} + \frac{1}{8} - \frac{1}{32} + \frac{1}{128} - \cdots$

$a = 2, \quad r = -\frac{1}{4}$

So the series converges to $\dfrac{2}{1+1/4} = \dfrac{2}{5/4} = 8/5$

(f) $2 - 1 + 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \cdots$

$S_n = (2-1) + (1 - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + \cdots + (\frac{2}{n} - \frac{2}{n+1})$

$= 2 - \frac{2}{n+1} \longrightarrow 2$ as $n \to \infty$

(g) $\displaystyle\sum_{n=3}^{\infty}\left(\frac{1}{n} - \frac{1}{n+2}\right) = \sum_{k=3}^{\infty}\left(\frac{1}{k} - \frac{1}{k+2}\right)$

$S_3 = \frac{1}{3} - \frac{1}{5}$

$S_4 = \left(\frac{1}{3} - \frac{1}{5}\right) + \left(\frac{1}{4} - \frac{1}{6}\right)$

$\vdots$

$S_n = \left(\frac{1}{3} - \cancel{\frac{1}{5}}\right) + \left(\frac{1}{4} - \cancel{\frac{1}{6}}\right) + \left(\cancel{\frac{1}{5}} - \cancel{\frac{1}{7}}\right) + \left(\cancel{\frac{1}{6}} - \cancel{\frac{1}{8}}\right) + \cdots + \left(\cancel{\frac{1}{n-1}} - \frac{1}{n+1}\right) + \left(\cancel{\frac{1}{n}} - \frac{1}{n+2}\right)$

$= \frac{1}{3} + \frac{1}{4} - \frac{1}{n+1} - \frac{1}{n+2} \longrightarrow \frac{1}{3} + \frac{1}{4}$ as $n \to \infty$

$\displaystyle\sum_{k=3}^{\infty}\left(\frac{1}{k} - \frac{1}{k+2}\right) = \frac{1}{3} + \frac{1}{4} = \frac{7}{12}$

(h) $\sum_{n=0}^{\infty}\left(e^{\frac{1}{n}} - e^{\frac{1}{n+1}}\right)$ $= \sum_{k=0}^{\infty}\left(e^{1/k} - e^{\frac{1}{k+1}}\right)$

$$S_n = \left(e^{1} - \cancel{e^{\frac{1}{2}}}\right) + \left(\cancel{e^{1/2}} - \cancel{e^{1/3}}\right) + \left(\cancel{e^{1/3}} - \cancel{e^{1/4}}\right) + \cdots + \left(\cancel{e^{\frac{1}{n}}} - e^{\frac{1}{n+1}}\right)$$

$$S_n = e^{1} - e^{\frac{1}{n+1}}$$

$$\lim_{n\to\infty} S_n = \left(e^{1} - e^{0}\right) = e - 1$$

Note: n has to start from 1. As is the series is not defined correctly.

(i) $\sum_{n=1}^{\infty}\frac{2}{n^2+n}$ $\quad \frac{2}{k^2+k} = \frac{2}{k(k+1)} = \frac{A}{k} + \frac{B}{k+1}$

$$A(k+1) + Bk = 2$$

$$\begin{cases} A + B = 0 \\ A = 2 \end{cases} \Rightarrow B = -2$$

$$\frac{2}{k^2+k} = \frac{2}{k} - \frac{2}{k+1}$$

$$S_n = \sum_{k=1}^{n}\left(\frac{2}{k} - \frac{2}{k+1}\right) = \left(2 - \cancel{\tfrac{2}{2}}\right) + \left(\cancel{\tfrac{2}{2}} - \cancel{\tfrac{2}{3}}\right) + \left(\cancel{\tfrac{2}{3}} - \cancel{\tfrac{2}{4}}\right) + \left(\cancel{\tfrac{2}{4}} - \tfrac{2}{5}\right) + \cdots$$

$$+ \left(\cancel{\tfrac{2}{n}} - \tfrac{2}{n+1}\right)$$

$$= 2 - \frac{2}{n+1} \to 2 - 0$$

$$= 2$$

(j) $\sum_{n=2}^{\infty} \frac{6n+3}{n^4+2n^3+n^2}$

$$\frac{6n+3}{n^4+2n^3+n^2} = \frac{6n+3}{n^2(n^2+2n+1)}$$

$$= \frac{6n+3}{n^2(n+1)^2}$$

$$\frac{A}{n} + \frac{B}{n^2} + \frac{C}{n+1} + \frac{D}{(n+1)^2} = \frac{6n+3}{n^2(n+1)^2}$$

$$An(n+1)^2 + B(n+1)^2 + C(n+1)n^2 + Dn^2 = 6n+3$$

$n=0$ gives us

$$B = 3$$

$n=-1$ gives us

$$D = -3$$

$n=1$ gives us

$$4A + 4B + 2C + D = 9$$

But $B=3$ and $D=-3$

$$\Rightarrow \quad 4A + 2C = 9 - 12 + 3 = 0$$

i.e, $4A + 2C = 0$

$n=2$ gives us

$$18A + 9B + 12C + 4D = 15$$

$$\Rightarrow 18A + 12C = 15 - 9(3) - 4(-3) = 15 - 27 + 12 = 0$$

i.e $18A + 12C = 0$

$$\begin{cases} 4A + 2C = 0 \\ 18A + 12C = 0 \end{cases} \Rightarrow \begin{cases} 2A + C = 0 \\ 3A + 2C = 0 \end{cases}$$ This gives $A = 0 = C$

So $\sum_{n=2}^{\infty} \frac{6n+3}{n^2(n+1)^2} = \sum_{n=2}^{\infty} \frac{3}{n^2} - \frac{3}{(n+1)^2}$ which converges to $\frac{3}{4}$.