Worksheet 11.9, 11.10 and 11.11

Full Name: ______________________________ Score: ________

1. Find the power series representation for each function f. Include the open interval of convergence.

   (a) $f(x) = \dfrac{x^2}{1-3x}$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad |x|<1$$

$$\frac{1}{1-3x} = \sum_{n=0}^{\infty} (3x)^n, \quad |3x|<1$$

$$\frac{x^2}{1-3x} = \sum_{n=0}^{\infty} x^2 \cdot 3^n \cdot x^n \qquad |x|<\frac{1}{3}$$

$$\frac{x^2}{1-3x} = \sum_{n=0}^{\infty} 3^n x^{n+2}, \quad |x|<\frac{1}{3}$$

$$\left(-\frac{1}{3}, \frac{1}{3}\right)$$

   (b) $f(x) = \dfrac{8}{4+7x} = \dfrac{8}{4(1+\frac{7}{4}x)} = \dfrac{2}{1+\frac{7}{4}x}$

$$\frac{1}{1+\frac{7}{4}x} = \sum_{n=0}^{\infty} \underbrace{\left(-\frac{7}{4}x\right)^n}_{=(-1)^n\left(\frac{7}{4}\right)^n\cdot x^n} \qquad \left|-\frac{7}{4}x\right|<1$$

$$\frac{2}{1+\frac{7}{4}x} = \sum_{n=0}^{\infty} (-1)^n \cdot 2 \cdot \frac{7^n}{4^n} x^n, \qquad |x|<\frac{4}{7}$$

so, $$\frac{2}{1+\frac{7}{4}x} = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{7^n}{2^{2n-1}} \cdot x^n,$$

$$|x|<\frac{4}{7}$$

$$\left(-\frac{4}{7}, \frac{4}{7}\right)$$

   (c) $f(x) = \dfrac{1}{(1-x^2)^2}$

$$\frac{1}{(1-x)^2} = \frac{d}{dx}\left(\frac{1}{1-x}\right) = \sum_{n=1}^{\infty} n x^{n-1}, \quad |x|<1$$

$$\frac{1}{(1-x^2)^2} = \sum_{n=1}^{\infty} n (x^2)^{n-1}, \qquad |x|<1$$

$$\frac{1}{(1-x^2)^2} = \sum_{n=1}^{\infty} n x^{2n-2}, \quad |x|<1$$

$$(-1, 1)$$

(d) $f(x) = \ln|2+3x|$

$$\frac{1}{2+3x} = \frac{1}{2(1+\frac{3}{2}x)} = \frac{1}{2}\cdot\frac{1}{1+\frac{3}{2}x} = \frac{1}{2}\sum_{n=0}^{\infty}\underbrace{\left(\frac{3}{2}x\right)^n}_{\frac{3^n x^n}{2^n}}, \quad \left|\frac{3}{2}x\right|<1$$

$$\frac{1}{2+3x} = \sum_{n=0}^{\infty}\frac{3^n x^n}{2\cdot 2^n}, \quad |x|<\frac{2}{3}$$

$$\ln|2+3x| = C + 3\cdot\int\frac{1}{2+3x}\,dx = C + 3\cdot\sum_{n=0}^{\infty}\frac{3^n}{2^{n+1}}\cdot\frac{1}{n+1}x^{n+1}$$

$x=0,$

$\ln 2 = C + 0$

$C = \ln 2$

$$\ln|2+3x| = \ln 2 + \sum_{n=0}^{\infty}\frac{3^{n+1}}{2^{n+1}}x^{n+1},$$

$|x|<\frac{2}{3}$

(e) $f(x) = xe^{-3x}$

$$e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!}$$

$$e^{-3x} = \sum_{n=0}^{\infty}\frac{(-3x)^n}{n!}, \quad (-\infty,\infty)$$

$$xe^{-3x} = \sum_{n=0}^{\infty}\frac{(-1)^n\cdot 3^n\cdot x\cdot x^n}{n!}$$

$$xe^{-3x} = \sum_{n=0}^{\infty}\frac{(-1)^n 3^n}{n!}x^{n+1}$$

(f) $f(x) = \cos(4x)$

$$\cos x = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{2n}$$

$$\cos 4x = \sum_{n=0}^{\infty}\frac{(-1)^n\cdot(4x)^{2n}}{(2n)!} = \sum_{n=0}^{\infty}\frac{(-1)^n 4^{2n}}{(2n)!}x^{2n}$$

(g) $f(x) = 4\sin(x^2)$

$$\sin(x^2) = \sum_{n=0}^{\infty}\frac{(-1)^n (x^2)^{2n+1}}{(2n+1)!}$$

$$4\sin(x^2) = \sum_{n=0}^{\infty}\frac{(-1)^n\cdot 4\cdot x^{4n+2}}{(2n+1)!}$$

2. Find the degree 3 Taylor polynomial centered at $a = 2$ for the function given by $f(x) = 2x^3 - 7x + 12$. Then multiply out and simplify your answer. Does the result surprise you?

$T_3(x) = C_0 + C_1 x + C_2 x^2 + C_3 x^3$

$f(x) = 2x^3 - 7x + 12$, $f(2) = 14$, $C_0 = 14$

$f'(x) = 6x^2 - 7$, $f'(2) = 17$, $C_1 = 17$

$f''(x) = 12x$ $f''(2) = 24$, $C_2 = \frac{24}{2!} = 12$

$f'''(x) = 12$, $f'''(2) = 12$, $C_3 = \frac{12}{3!} = 2$

$T_3(x) = 14 + 17(x-2) + 12(x-2)^2 + 2(x-2)^3$

3. A car is moving with speed 20 m/s and acceleration 2 m/s^2 at a given instant. Using a second-degree Taylor polynomial, estimate how far the car moves in the next second. Would it be reasonable to use this polynomial to estimate the distance traveled during the next minute?

$v = 20 m/s$
$a = 2 m/s^2$

$s(0) = 0$

$s'(0) = 20$

$s''(0) = 2$

$s(t) \approx s(0) + s'(0)t + s''(0) \cdot t^2$

$s(t) \approx 20t + 2t^2$

$s(1) \approx 20 + 2 = 22$

4. Find the degree 0, degree 1, degree 2, degree 3, and degree 4 Taylor polynomials centered at $a = 9$ for the function given by $f(x) = \sqrt{x}$. Use each of these to get successively better approximations for $\sqrt{10}$.

$f(x) = \sqrt{x}$ $\quad f(9) = 3$ $\quad c_0 = 3$

$f'(x) = \frac{1}{2}x^{-1/2}$ $\quad f'(9) = \frac{1}{3 \cdot 3} = \frac{1}{9}$ $\quad c_1 = \frac{1}{9}$

$f''(x) = -\frac{1}{4}x^{-3/2}$ $\quad f''(9) = \frac{-1}{4\sqrt{9^3}} = \frac{-1}{4 \cdot 27} = \frac{-1}{108}$ $\quad c_2 = \frac{-1/108}{2!} = \frac{-1}{216}$

$f'''(x) = \frac{3}{8}x^{-5/2}$ $\quad f'''(9) = \frac{3}{8\sqrt{9^5}} = \frac{1}{648}$ $\quad c_3 = \frac{1/648}{3!} = \frac{1}{3888}$

$f^{(4)}(x) = \frac{-15}{16}x^{-7/2}$ $\quad f^{(4)}(9) = \frac{-15}{16\sqrt{9^7}} = \frac{-5}{11664}$ $\quad c_4 = \frac{-5/11664}{4!} = \frac{-5}{279936}$

$T_0(x) = 3$

$T_1(x) = 3 + \frac{1}{9}(x-9)$

$T_2(x) = 3 + \frac{1}{9}(x-9) - \frac{1}{216}(x-9)^2$

$T_3(x) = 3 + \frac{1}{9}(x-9) - \frac{1}{216}(x-9)^2 + \frac{1}{3888}(x-9)^3$

$T_4(x) = 3 + \frac{1}{9}(x-9) - \frac{1}{216}(x-9)^2 + \frac{1}{3888}(x-9)^3 - \frac{5}{279936}(x-9)^4$