0.1 U-Substitution

We make a u-substitution to fill in the gaps in the equation

$$\int f(g(x))g'(x)\ dx = F(g(x)) + C.$$

If we let

$$u = g(x)$$

then

$$du = g'(x)\ dx$$

and the original integral can be rewritten as

$$\int f(g(x))g'(x)\ dx = \int f(u)\ du.$$

We can now try to find an anti-derivative of $f(u)$:

$$\int f(u)\ du = F(u) + C$$

and back substituting for u gives

$$F(u) + C = F(g(x)) + C.$$

Now we can conclude that

$$\int f(g(x))g'(x)\ dx = F(g(x)) + C.$$

Examples of U-Substitution

US 1. To compute $\int 2x\cos(x^2+1)\ dx$, we let $u = x^2+1$. We have,

$$u = x^2 + 1$$
$$du = 2x\ dx$$

and the integral can be written as

$$\int 2x\cos(x^2+1)\ dx = \int \cos(x^2+1)\ 2x\ dx = \int \cos(u)\ du.$$

The last integral can be computed as

$$\int \cos(u)\ du = \sin(u) + C$$

and by back substituting, we have

$$\sin(u) + C = \sin(x^2+1) + C.$$

Thus, using u-substitution we can conclude that

$$\int 2x\cos(x^2+1)\ dx = \sin(x^2+1) + C.$$

US 2. To compute $\int 2x(x^2+1)^4\ dx$, we let $u = x^2+1$. We have,

$$u = x^2 + 1$$
$$du = 2x\ dx$$

and the integral can be written as

$$\int 2x(x^2+1)^4\ dx = \int (x^2+1)^4\ 2x\ dx = \int u^4\ du.$$

The last integral can be computed as

$$\int u^4 \, du = \frac{u^5}{5} + C$$

and by back substituting, we have

$$\frac{u^5}{5} + C = \tfrac{1}{5}(x^2+1)^5 + C.$$

Thus, using u-substitution we can conclude that

$$\int 2x(x^2+1)^4 \, dx = \tfrac{1}{5}(x^2+1)^5 + C.$$

US 3. To compute $\displaystyle\int 2xe^{(x^2+1)} \, dx$, we let $u = x^2+1$. We have,

$$u = x^2 + 1$$

$$du = 2x \, dx$$

$$\int 2xe^{(x^2+1)} \, dx = \int e^{(x^2+1)} \, 2x \, dx = \int e^u \, du.$$

The last integral can be computed as

$$\int e^u \, du = e^u + C$$

and by back substituting, we have

$$e^u + C = e^{(x^2+1)} + C.$$

Thus, using u-substitution we can conclude that

$$\int 2xe^{(x^2+1)} \, dx = e^{(x^2+1)} + C.$$

US 4. To compute $\int 2x\sqrt{x^2+1}\ dx$, we let $u = x^2+1$. We have,

$$u = x^2 + 1$$
$$du = 2x\ dx$$

$$\int 2x\sqrt{x^2+1}\ dx = \int \sqrt{x^2+1} \cdot 2x\ dx = \int \sqrt{u}\ du.$$

The last integral can be computed as

$$\int \sqrt{u}\ du = \int u^{1/2}\ du = \frac{u^{3/2}}{3/2} + C = \tfrac{2}{3}u^{3/2} + C$$

and by back substituting, we have

$$\frac{2}{3}u^{3/2} + C = \frac{2}{3}(x^2+1)^{3/2} + C.$$

Thus, using u-substitution we can conclude that

$$\int 2x\sqrt{x^2+1}\ dx = \tfrac{2}{3}(x^2+1)^{3/2} + C.$$

US 5. To compute $\int \frac{2x}{x^2+1}\ dx$, we let $u = x^2+1$. We have,

$$u = x^2 + 1$$
$$du = 2x\ dx$$

$$\int \frac{2x}{x^2+1}\ dx = \int \frac{1}{x^2+1} \cdot 2x\ dx = \int \frac{1}{u}\ du.$$

The last integral can be computed as

$$\int \frac{1}{u}\ du = \ln|u| + C$$

and by back substituting, we have

$$\ln|u| + C = \ln|x^2+1| + C = \ln(x^2+1) + C.$$

Thus, using u-substitution we can conclude that

$$\int \frac{2x}{x^2+1}\ dx = \ln(x^2+1) + C.$$

US 6. To compute $\displaystyle\int e^x \sin(e^x)\ dx$, we let $u = e^x$. We have,

$$u = e^x$$

$$du = e^x\ dx$$

and the integral can be written as

$$\int e^x \sin(e^x)\ dx = \int \sin(e^x)\cdot e^x\ dx = \int \sin(u)\ du.$$

The last integral can be computed as

$$\int \sin(u)\ du = -\cos(u) + C$$

and by back substituting, we have

$$-\cos(u) + C = -\cos(e^x) + C.$$

Thus, using u-substitution we can conclude that

$$\int e^x \sin(e^x)\ dx = -\cos(e^x) + C.$$

US 7. To compute $\displaystyle\int 3x^2 \sec(x^3)\tan(x^3)\ dx$, we let $u = x^3$. We have,

$$u = x^3$$

$$du = 3x^2 \ dx$$

and the integral can be written as

$$\begin{aligned}\int 3x^2 \sec(x^3)\tan(x^3) \ dx &= \int \sec(x^3)\tan(x^3)\cdot 3x^2 \ dx \\ &= \int \sec(u)\tan(u) \ du.\end{aligned}$$

The last integral can be computed as

$$\int \sec(u)\tan(u) \ du = \sec(u) + C$$

and by back substituting, we have

$$\sec(u) + C = \sec(x^3) + C.$$

Thus, using u-substitution we can conclude that

$$\int 3x^2 \sec(x^3)\tan(x^3) \ dx = \sec(x^3) + C.$$

US 8. To compute $\int x^3 \cos(x^4) \ dx$, we let $u = x^4$. We have,

$$u = x^4$$

$$du = 4x^3 \ dx$$

and the integral can be written as

$$\begin{aligned}\int x^3 \cos(x^4) \ dx &= \int \cos(x^4)\cdot x^3 \ dx \\ &= \int \cos(x^4)\cdot \tfrac{1}{4}\cdot 4x^3 \ dx \\ &= \int \tfrac{1}{4}\cos(u) \ du.\end{aligned}$$

The last integral can be computed as

$$\int \tfrac{1}{4}\cos(u)\ du = \tfrac{1}{4}\sin(u) + C$$

and by back substituting, we have

$$\tfrac{1}{4}\sin(u) + C = \tfrac{1}{4}\sin(x^4) + C.$$

Thus, using u-substitution we can conclude that

$$\int x^3\cos(x^4)\ dx = \tfrac{1}{4}\sin(x^4) + C.$$

US 9. To compute $\displaystyle\int xe^{-x^2}\ dx$, we let $u = -x^2$. We have,

$$u = -x^2$$

$$du = -2x\ dx$$

and the integral can be written as

$$\begin{aligned} \int xe^{-x^2}\ dx &= \int e^{-x^2} \cdot x\ dx \\ &= \int e^{-x^2}(-\tfrac{1}{2})\ (-2x)\ dx \\ &= \int -\tfrac{1}{2}e^u\ du. \end{aligned}$$

The last integral can be computed as

$$\int -\tfrac{1}{2}e^u\ du = -\tfrac{1}{2}e^u + C$$

and by back substituting, we have

$$-\tfrac{1}{2}e^u + C = -\tfrac{1}{2}e^{-x^2} + C.$$

Thus, using u-substitution we can conclude that

$$\int xe^{-x^2}\ dx = -\tfrac{1}{2}e^{-x^2} + C.$$

US 10. To compute $\int \sin(3x)\ dx$, we let $u = 3x$. We have,

$$u = 3x$$

$$du = 3\ dx$$

and the integral can be written as

$$\int \sin(3x)\ dx = \int \sin(3x) \cdot \tfrac{1}{3} \cdot 3\ dx = \int \tfrac{1}{3} \sin(u)\ du.$$

The last integral can be computed as

$$\int \tfrac{1}{3} \sin(u)\ du = -\tfrac{1}{3} \cos(u) + C$$

and by back substituting, we have

$$-\tfrac{1}{3} \cos(u) + C = -\tfrac{1}{3} \cos(3x) + C.$$

Thus, using u-substitution we can conclude that

$$\int \sin(3x)\ dx = -\tfrac{1}{3} \cos(3x) + C.$$

US 11. To compute $\int e^{\frac{x}{2}}\ dx$, we let $u = \frac{x}{2}$. We have,

$$u = \frac{x}{2}$$

$$du = \tfrac{1}{2}\ dx$$

and the integral can be written as

$$\int e^{\frac{x}{2}}\ dx = \int e^{\frac{x}{2}} \cdot 2 \cdot \tfrac{1}{2}\ dx = \int 2e^u\ du.$$

The last integral can be computed as

$$\int 2e^u\ du = 2e^u + C$$

and by back substituting, we have

$$2e^u + C = 2e^{\frac{x}{2}} + C.$$

Thus, using u-substitution we can conclude that

$$\int e^{x/2}\, dx = 2e^{x/2} + C.$$

US 12. To compute $\int (4x+3)^5\, dx$, we let $u = 4x + 3$. We have,

$$u = 4x + 3$$

$$du = 4\, dx$$

and the integral can be written as

$$\int (4x+3)^5\, dx = \int (4x+3)^5 \cdot \tfrac{1}{4} \cdot 4\, dx = \int \tfrac{1}{4} u^5\, du.$$

The last integral can be computed as

$$\tfrac{1}{4} \int u^5\, du = \tfrac{1}{4} \cdot \frac{u^6}{6} + C = \frac{u^6}{24} + C$$

and by back substituting, we have

$$\frac{u^6}{24} + C = \frac{(4x+3)^6}{24} + C.$$

Thus, using u-substitution we can conclude that

$$\int (4x+3)^5\, dx = \tfrac{1}{24}(4x+3)^6 + C.$$

US 13. To compute $\displaystyle\int \frac{1}{x \ln^2(x)}\, dx$, we let $u = \ln(x)$. We have,

$$u = \ln(x)$$

$$du = \tfrac{1}{x}\ dx$$

and the integral can be written as

$$\int \frac{1}{x\ln^2(x)}\ dx = \int \frac{1}{\ln^2(x)} \cdot \frac{1}{x}\ dx = \int \frac{1}{u^2}\ du.$$

The last integral can be computed as

$$\int \frac{1}{u^2}\ du = \int u^{-2}\ du = \frac{u^{-1}}{-1} + C = -\frac{1}{u} + C$$

and by back substituting, we have

$$-\frac{1}{u} + C = -\frac{1}{\ln(x)} + C.$$

Thus, using u-substitution we can conclude that

$$\int \frac{1}{x\ln^2(x)}\ dx = -\frac{1}{\ln(x)} + C.$$

US 14. To compute $\displaystyle\int \sin^4(x)\cos(x)\ dx$, we let $u = \sin(x)$. We have,

$$u = \sin(x)$$

$$du = \cos(x)\ dx$$

and the integral can be written as

$$\int \sin^4(x)\cos(x)\ dx = \int u^4\ du.$$

The last integral can be computed as

$$\int u^4\ du = \frac{u^5}{5} + C$$

and by back substituting, we have

$$\frac{u^5}{5} + C = \tfrac{1}{5}\sin^5(x) + C.$$

Thus, using u-substitution we can conclude that

$$\int \sin^4(x)\cos(x)\ dx = \tfrac{1}{5}\sin^5(x) + C.$$

US 15. To compute $\displaystyle\int \tan(x)\ dx$, we first rewrite the integral:

$$\int \tan(x)\ dx = \int \frac{\sin(x)}{\cos(x)}\ dx.$$

Now, let $u = \cos(x)$; then $du = -\sin(x)\ dx$ and the integral can be written as

$$\begin{aligned}\int \frac{\sin(x)}{\cos(x)}\ dx &= \int \frac{1}{\cos(x)}\ \sin(x)\ dx\\ &= -\int \frac{1}{\cos(x)}\ \big(-\sin(x)\big)\ dx\\ &= -\int \frac{1}{u}\ du.\end{aligned}$$

The last integral can be computed as

$$-\int \frac{1}{u}\ du = -\ln|u| + C$$

and by back substituting, we have

$$-\ln|u| + C = -\ln|\cos(x)| + C = \ln|\sec(x)| + C.$$

Thus, using u-substitution we can conclude that

$$\int \tan(x)\ dx = \ln|\sec(x)| + C.$$

US 16. To compute the integral $\displaystyle\int \tan(x)\ dx$, we first rewrite the integral:

$$\int \tan(x)\ dx = \int \frac{\sin(x)}{\cos(x)}\ dx.$$

Now, let $u = \cos(x)$, then $du = -\sin(x)\ dx$ and the integral can be rewritten as

$$\int \frac{\sin(x)}{\cos(x)}\ dx = \int -\frac{1}{u}\ du.$$

The last integral can be computed as

$$\int -\frac{1}{u}\ du = -\ln|u| + C$$

and by back substituting, we have

$$-\ln|u| + C = -\ln|\cos(x)| + C = \ln|\sec(x)| + C.$$

Thus, using u-substitution we can conclude that

$$\int \tan(x)\ dx = \ln|\sec(x)| + C.$$

US 17. To compute the integral $\displaystyle\int \sec(x)\ dx$, we first rewrite the integral:

$$\int \sec(x)\ dx = \int \sec(x)\frac{\sec(x)+\tan(x)}{\sec(x)+\tan(x)}\ dx.$$

Distributing in the numerator, we get

$$\int \sec(x)\frac{\sec(x)+\tan(x)}{\sec(x)+\tan(x)}\ dx = \int \frac{\sec^2(x)+\sec(x)\tan(x)}{\sec(x)+\tan(x)}\ dx.$$

Now, let $u = \sec(x)+\tan(x)$, then $du = [\sec(x)\tan(x) + \sec^2(x)]\ dx$ and the integral can be rewritten as

$$\int \frac{\sec^2(x)+\sec(x)\tan(x)}{\sec(x)+\tan(x)}\ dx = \int \frac{1}{u}\ du.$$

The last integral can be computed as

$$\int \frac{1}{u}\ du = \ln|u| + C$$

and by back substituting, we have

$$\ln|u| + C = \ln|\sec(x) + \tan(x)| + C.$$

Thus, using u-substitution we can conclude that

$$\int \sec(x)\ dx = \ln|\sec(x) + \tan(x)| + C.$$