0.1 Product Rule

The Product Rule

$$[f(x) \cdot g(x)]' = f(x)g'(x) + g(x)f'(x)$$

In words, the derivative of a product is the first times the derivative of the second plus the second times the derivative of the first.

Examples of the Product Rule

PR 1. If $h(x) = x^3 \sin(x)$ then $h(x) = f(x) \cdot g(x)$ with:

$$f(x) = x^3 \text{ and } \ g(x) = \sin(x).$$

To use the product rule, we need the derivatives:

$$f'(x) = 3x^2 \text{ and } \ g'(x) = \cos(x).$$

We can now write:

$$\begin{aligned} h'(x) &= f(x)g'(x) + g(x)f'(x) \\ &= x^3 \cos(x) + \sin(x) \cdot 3x^2 \\ &= x^3 \cos(x) + 3x^2 \sin(x). \end{aligned}$$

PR 2. If $h(x) = 3x^5 \cos(x)$ then $h(x) = f(x) \cdot g(x)$ with:

$$f(x) = 3x^5 \text{ and } \ g(x) = \cos(x).$$

To use the product rule, we need the derivatives

$$f'(x) = 15x^4 \text{ and } \ g'(x) = -\sin(x).$$

We can now write:

$$\begin{aligned} h'(x) &= f(x)g'(x) + g(x)f'(x) \\ &= 3x^5(-\sin(x)) + \cos(x) \cdot 15x^4 \\ &= -3x^5 \sin(x) + 15x^4 \cos(x) \\ &= 15x^4 \cos(x) - 3x^5 \sin(x). \end{aligned}$$

PR 3. If $h(x) = (x^2 - 5x - 2)e^x$ then $h(x) = f(x) \cdot g(x)$ with:

$$f(x) = x^2 - 5x - 2 \text{ and } \ g(x) = e^x.$$

To use the product rule, we need the derivatives:

$$f'(x) = 2x - 5 \text{ and } \ g'(x) = e^x.$$

We can now write:

$$\begin{aligned} h'(x) &= f(x)g'(x) + g(x)f'(x) \\ &= (x^2 - 5x - 2)e^x + e^x(2x - 5) \\ &= (x^2 - 5x - 2 + 2x - 5)e^x \\ &= (x^2 - 3x - 7)e^x. \end{aligned}$$

PR 4. If $h(x) = x\ln(x)$ then $h(x) = f(x) \cdot g(x)$ with:

$$f(x) = x \text{ and } \ g(x) = \ln(x).$$

To use the product rule, we need the derivatives:

$$f'(x) = 1 \text{ and } \ g'(x) = \frac{1}{x}.$$

We can now write

$$\begin{aligned} h'(x) &= f(x)g'(x) + g(x)f'(x) \\ &= x \cdot \frac{1}{x} + \ln(x) \cdot 1 \\ &= 1 + \ln(x). \end{aligned}$$

PR 5. If $h(x) = e^x \sin(x)$ then $h(x) = f(x) \cdot g(x)$ with:

$$f(x) = e^x \text{ and } \ g(x) = \sin(x).$$

To use the product rule, we need the derivatives:

$$f'(x) = e^x \text{ and } \ g'(x) = \cos(x).$$

We can now write:

$$\begin{aligned} h'(x) &= f(x)g'(x) + g(x)f'(x) \\ &= e^x \cos(x) + \sin(x)e^x \\ &= e^x[\sin(x) + \cos(x)]. \end{aligned}$$

PR 6. If $h(x) = (1 + x^2)\tan^{-1}(x)$ then $h(x) = f(x) \cdot g(x)$ with:

$$f(x) = 1 + x^2 \text{ and } \; g(x) = \tan^{-1}(x).$$

To use the product rule we need the derivatives:

$$f'(x) = 2x \text{ and } \; g'(x) = \frac{1}{1 + x^2}.$$

We can now write:

$$\begin{aligned} h'(x) &= f(x)g'(x) + g(x)f'(x) \\ &= (1 + x^2)\frac{1}{1 + x^2} + \tan^{-1}(x) \cdot 2x \\ &= 1 + 2x\tan^{-1}(x). \end{aligned}$$

0.2 Quotient Rule

The Quotient Rule

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{g(x)f'(x) - f(x)g'(x)}{g^2(x)}$$

In words, the derivative of a quotient is the bottom times the derivative of the top minus the top times the derivative of the bottom, all over the bottom squared.

Examples of the Quotient Rule

QR 1. If $h(x) = \dfrac{x+1}{x-1}$ then $h(x) = \dfrac{f(x)}{g(x)}$ with:

$$f(x) = x + 1 \text{ and } \; g(x) = x - 1.$$

To use the quotient rule we need the derivatives:

$$f'(x) = 1 \text{ and } \; g'(x) = 1.$$

We can now write:

$$\begin{aligned} h'(x) &= \frac{g(x)f'(x) - f(x)g'(x)}{g^2(x)} \\ &= \frac{(x-1)(1) - (x+1)(1)}{(x-1)^2} \\ &= \frac{(x-1) - (x+1)}{(x-1)^2} \\ &= -\frac{2}{(x-1)^2}. \end{aligned}$$

QR 2. If $h(x) = \dfrac{x^2 - 3x}{x^3 + 5}$ then $h(x) = \dfrac{f(x)}{g(x)}$ with:

$$f(x) = x^2 - 3x \text{ and } \; g(x) = x^3 + 5.$$

To use the quotient rule we need the derivatives:

$$f'(x) = 2x - 3 \text{ and } \; g'(x) = 3x^2.$$

We can now write:

$$\begin{aligned} h'(x) &= \frac{g(x)f'(x) - f(x)g'(x)}{g^2(x)} \\ &= \frac{(x^3+5)(2x-3) - (x^2-3x)(3x^2)}{(x^3+5)^2} \\ &= \frac{(2x^4 - 3x^3 + 10x - 15) - (3x^4 - 9x^3)}{(x^3+5)^2} \\ &= \frac{2x^4 - 3x^3 + 10x - 15 - 3x^4 + 9x^3}{(x^3+5)^2} \\ &= \frac{-x^4 + 6x^3 + 10x - 15}{(x^3+5)^2} \end{aligned}$$

QR 3. If $h(x) = \dfrac{\cos(x)}{2x + 1}$ then $h(x) = \dfrac{f(x)}{g(x)}$ with:

$$f(x) = \cos(x) \text{ and } \; g(x) = 2x + 1.$$

To use the quotient rule we need the derivatives:

$$f'(x) = -\sin(x) \text{ and } \; g'(x) = 2.$$

We can now write:

$$\begin{aligned} h'(x) &= \frac{g(x)f'(x) - f(x)g'(x)}{g^2(x)} \\ &= \frac{(2x+1)(-\sin(x)) - \cos(x) \cdot 2}{(2x+1)^2} \\ &= -\frac{(2x+1)\sin(x) + 2\cos(x)}{(2x+1)^2} \end{aligned}$$

QR 4. If $h(x) = \dfrac{e^x}{x^2+1}$ then $h(x) = \dfrac{f(x)}{g(x)}$ with:

$$f(x) = e^x \text{ and } \ g(x) = x^2 + 1.$$

To use the quotient rule we need the derivatives:

$$f'(x) = e^x \text{ and } \ g'(x) = 2x.$$

We can now write:

$$\begin{aligned} h'(x) &= \frac{g(x)f'(x) - f(x)g'(x)}{g^2(x)} \\ &= \frac{(x^2+1)e^x - e^x(2x)}{(x^2+1)^2} \\ &= \frac{(x^2-2x+1)e^x}{(x^2+1)^2} \\ &= \frac{(x-1)^2e^x}{(x^2+1)^2} \end{aligned}$$

QR 5. If $h(x) = \dfrac{\ln(x)}{x+1}$ then $h(x) = \dfrac{f(x)}{g(x)}$ with:

$$f(x) = \ln(x) \text{ and } \ g(x) = x + 1.$$

To use the quotient rule we need the derivatives:

$$f'(x) = 1/x \text{ and } \ g'(x) = 1.$$

We can now write:

$$\begin{aligned} h'(x) &= \frac{g(x)f'(x) - f(x)g'(x)}{g^2(x)} \\ &= \frac{(x+1)(1/x) - \ln(x)}{(x+1)^2} \\ &= \frac{(x+1)(1/x) - \ln(x)}{(x+1)^2} \cdot \frac{x}{x} \\ &= \frac{(x+1) - x\ln(x)}{x(x+1)^2}. \end{aligned}$$

QR 6. If $h(x) = \tan(x)$, we first rewrite $h(x)$ as $\dfrac{\sin(x)}{\cos(x)}$ and then $h(x) = \dfrac{f(x)}{g(x)}$ with:

$$f(x) = \sin(x) \text{ and } \; g(x) = \cos(x).$$

To use the quotient rule we need the derivatives:

$$f'(x) = \cos(x) \text{ and } \; g'(x) = -\sin(x).$$

We can now write:

$$\begin{aligned} h'(x) &= \frac{g(x)f'(x) - f(x)g'(x)}{g^2(x)} \\ &= \frac{\cos(x)\cos(x) - \sin(x)(-\sin(x))}{\cos^2(x)} \\ &= \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)} \\ &= \frac{1}{\cos^2(x)} = \sec^2(x) \end{aligned}$$

This important formula is worth remembering: if $f(x) = \tan(x)$ then $f'(x) = \sec^2(x)$.

QR 7. If $h(x) = \cot(x)$, we first rewrite $h(x)$ as $\dfrac{\cos(x)}{\sin(x)}$ and then $h(x) = \dfrac{f(x)}{g(x)}$ with:

$$f(x) = \cos(x) \text{ and } \; g(x) = \sin(x).$$

To use the quotient rule we need the derivatives:

$$f'(x) = -\sin(x) \text{ and } \; g'(x) = \cos(x).$$

We can now write:

$$\begin{aligned} h'(x) &= \frac{g(x)f'(x) - f(x)g'(x)}{g^2(x)} \\ &= \frac{\sin(x)(-\sin(x)) - \cos(x)\cos(x)}{\sin^2(x)} \\ &= -\frac{\sin^2(x) + \cos^2(x)}{\sin^2(x)} \\ &= -\frac{1}{\sin^2(x)} = -\csc^2(x) \end{aligned}$$

This important formula is worth remembering: if $f(x) = \cot(x)$ then $f'(x) = -\csc^2(x)$.

QR 8. If $h(x) = \sec(x)$ first we rewrite $h(x)$ as $\dfrac{1}{\cos(x)}$ and then $h(x) = \dfrac{f(x)}{g(x)}$ with:

$$f(x) = 1 \text{ and } \; g(x) = \cos(x).$$

To use the quotient rule we need the derivatives:

$$f'(x) = 0 \text{ and } \; g'(x) = -\sin(x).$$

We can now write:

$$\begin{aligned} h'(x) &= \frac{g(x)f'(x) - f(x)g'(x)}{g^2(x)} \\ &= \frac{\cos(x) \cdot 0 - 1 \cdot (-\sin(x))}{\cos^2(x)} \\ &= \frac{\sin(x)}{\cos^2(x)} \\ &= \frac{1}{\cos(x)} \cdot \frac{\sin(x)}{\cos(x)} \\ &= \sec(x)\tan(x) \end{aligned}$$

This important formula is worth remembering: if $f(x) = \sec(x)$ then $f'(x) = \sec(x)\tan(x)$.

QR 9. The last trig function is $\csc(x)$ and, if $f(x) = \csc(x)$ then $f'(x) = -\csc(x)\cot(x)$.

QR 10. If $h(x) = \dfrac{x^2 + 3x + 5}{x}$ then we could find $h'(x)$ by using the quotient rule, but it is easier to rewrite the functions as $h(x) = x + 3 + \frac{5}{x} = x + 3 + 5x^{-1}$ and find $h'(x)$ using the constant rule, power rule and the constant multiple rule. We have:

$$h'(x) = 1 + (-5)x^{-2} = 1 - \frac{5}{x^2}.$$