

0.1 Optimization

In this section, we find maximums and minimums of functions as required by real life situations.

Examples of Optimization

OZ 1. Farmer Bob has 2400 linear feet of fence with which to build a rectangular enclosure with a vertical partition. What dimensions will maximize the total area covered by the enclosure and what is the maximum area?

Let x be the length of the enclosure and y the width. The total area of the enclosure is then

$$A = xy.$$

This is called the *objective* function. To maximize the area, Farmer Bob should use all 2400 linear feet of fence. Since there are two horizontal strips of length x and three vertical strips of length y (b/c of the partition), we get the *constraint* equation:

$$2x + 3y = 2400.$$

In the standard language of optimization, we can now state the problem as follows.

Maximize the objective

$$A = xy,$$

subject to the constraint

$$2x + 3y = 2400.$$

To find the maximum area, we need to eliminate one of the variables x, y from the objective function. We use the constraint to do this. Solve the constraint for either x or y , whichever is easier. In this case, solving the constraint for y gives

$$y = 800 - \frac{2}{3}x.$$

We now *substitute* this expression of y into the objective function to get

$$A = xy = x(800 - \frac{2}{3}x) = 800x - \frac{2}{3}x^2.$$

Now we can find the maximum area by finding the critical numbers:

$$A' = 800 - \frac{4}{3}x = 0$$

which yields

$$x = 600 \text{ ft.}$$

Note that this gives a maximum since the graph of A is a parabola that opens down. To finish off the problem, we find y by plugging $x = 600$ into the constraint, which we already solved for y :

$$y = 800 - \frac{2}{3}x = 800 - \frac{2}{3}(600) = 400 \text{ ft.}$$

Finally the maximum area of the enclosure is

$$A = xy = (600)(400) = 240,000 \text{ sq.ft.}$$

OZ 2. Gardner Harold wants to construct a 1600 square foot rectangular enclosure that has both a horizontal and a vertical partition. What dimensions will require the minimum amount of fencing? How much fence will he need?

Let x be the length of the enclosure and y the width. Because of the partitions, there are 3 horizontal and 3 vertical sections of fence. Hence the total amount of fence needed can be expressed as

$$F = 3x + 3y.$$

This is our objective function. Since the area of the enclosure needs to be 1600 sq. ft., we have the constraint

$$xy = 1600.$$

Solving the constraint for y gives $y = 1600/x$. Substituting this into the objective yields

$$F = 3x + 3y = 3x + 3\left(\frac{1600}{x}\right) = 3x + \frac{4800}{x}.$$

The minimum value of F occurs when $F' = 0$. Solving $F' = 0$ for x gives

$$F' = 3 - \frac{4800}{x^2} = 0$$

which means $3x^2 = 4800$ and so $x = \pm 40$ ft. Since $x > 0$ from context, we have $x = 40$ ft. as our solution. Plugging this value of x into the constraint equation gives

$$y = \frac{1600}{x} = \frac{1600}{40} = 40 \text{ ft.}$$

So the solution is for the gardener to build a square, 40 feet on a side, using a total of $F = 3x + 3y = 3(40) + 3(40) = 240$ feet of fence.

OZ3 Scientist Sam wants to know how close a comet moving in a parabolic trajectory will get to the sun. We will assume that the sun is located at the origin, the path of the comet follows the parabola $y = x^2 - 5$ and that the units on the axes are in millions of miles.

Using the distance formula, we can see that if the comet is at a point (x, y) , then its distance to the sun at $(0, 0)$ is

$$d = \sqrt{x^2 + y^2}.$$

Since the comet is on the parabola $y = x^2 - 5$, we can substitute this into the equation for d giving

$$d = \sqrt{x^2 + (x^2 - 5)^2}.$$

Now, we would like to minimize d . A convenient trick when minimizing a square root is to minimize the radicand. So, let $f(x) = x^2 + (x^2 - 5)^2$. We will minimize $f(x)$. The min occurs when the derivative is 0, so we solve $f'(x) = 0$ for x yielding

$$f'(x) = 2x + 2(x^2 - 5)(2x) = 2x[1 + 2(x^2 - 5)] = 2x(2x^2 - 9) = 0$$

which gives either

$$2x = 0 \text{ or } 2x^2 - 9 = 0.$$

The three solutions are $x = 0, \pm 3/\sqrt{2}$. The corresponding y -values can be found by plugging these x -values into the parabola $y = x^2 - 5$ giving the three points

$$(0, -5), (-3/\sqrt{2}, -1/2) \text{ and } (3/\sqrt{2}, -1/2).$$

If the comet is at $(0, -5)$ then its distance to the sun is 5 million miles. If the comet is at either $(\pm 3/\sqrt{2}, -1/2)$ then the distance is

$$d = \sqrt{x^2 + y^2} = \sqrt{\frac{9}{2} + \frac{1}{4}} = \frac{\sqrt{19}}{2} \text{ million miles.}$$

Hence the minimum distance is $\sqrt{19}/2$ million miles when the comet is at either of the points $(\pm 3/\sqrt{2}, -1/2)$.

OZ 4. A box with a square base and an open top are to be constructed using 4800 sq. in. of cardboard. Find the dimensions of the box that will maximize its volume. What is the maximum volume?

If we let x represent the length and width of the box, and y its height, then our objective is to maximize the volume,

$$V = \text{length} \times \text{width} \times \text{height} = x^2 y.$$

We have a material constraint which says that the surface area of the box should be 4800 sq. in:

$$\text{Area of base} + \text{area of 4 sides} = x^2 + 4xy = 4800.$$

Solving the constraint for y gives

$$y = \frac{4800 - x^2}{4x},$$

and substituting this into the objective gives:

$$V = x^2 y = x^2 * \frac{4800 - x^2}{4x}.$$

This simplifies to

$$V = 1200x - \frac{1}{4}x^3$$

which has a maximum when its derivative is zero. Solving $V' = 0$ for x gives

$$V' = 1200 - \frac{3}{4}x^2 = 0,$$

so

$$\frac{3}{4}x^2 = 1200,$$

and

$$\begin{aligned}x^2 &= 1600, \\x &= \pm 40,\end{aligned}$$

so our answer is

$$x = 40 \text{ in.}$$

Plugging this into the constraint equation, we can find y :

$$y = \frac{4800 - x^2}{4x} = \frac{4800 - 1600}{160} = 20 \text{ in.}$$

The volume of the box with these dimensions is

$$V = x^2 y = (40)^2(20) = 32,000 \text{ cubic inches.}$$