1 Finding Limits Numerically

In this section, functions will be presented using formulas. We will determine the limit of a function by making an appropriate **table of values**.

NL 1. Determine the following limit:

$$\lim_{x \to 3^+} x^2 + 2.$$

First, $x \to 3^+$ means that x is near the number 3 and $x > 3$. To conceptualize this numerically, consider the following pattern of numbers: $3.1, 3.01, 3.001, 3.0001$.

To determine the limit numerically, we will plug these four values of x into the function and look for a pattern in the outputs.

x	3.1	3.01	3.001	3.0001
$x^2 + 2$	11.61	11.0601	11.006001	11.00060001

The outputs appear to be approaching the number 11, so we write

$$\lim_{x \to 3^+} x^2 + 2 = 11.$$

In this example, we could have arrived at the answer by simply plugging in $x = 3$. In general, when plugging in the value that x is approaching yields a reasonable answer, that answer is usually correct. We now turn our attention to examples where this shortcut does not work.

NL 2. Compute the following limit:

$$\lim_{x \to 3^-} \frac{|x-3|}{x-3}.$$

As in the previous example, x is near 3 but this time, $x < 3$. We therefore use the following numerical values for x: $2.9, 2.99, 2.999, 2.9999$ to generate the following table of values:

x	2.9	2.99	2.999	2.9999
$\frac{\lvert x-3 \rvert}{x-3}$	-1	-1	-1	-1

The values of the function are not just approaching -1, they actually are -1 and so we can conclude that

$$\lim_{x \to 3^-} \frac{|x-3|}{x-3} = -1.$$

It should be noticed that the shortcut of plugging in $x = 3$ yields the meaningless **indeterminate form** $0/0$.

The next example has far-reaching consequences that will be exploited in the next chapter.

NL 3. Compute the following limit:

$$\lim_{x\to 0^+} \frac{e^x - 1}{x}.$$

In this example, x is near 0 and $x > 0$. The values of x we will use to generate a table of values are: $.1, .01, .001$ and $.0001$.

x	0.1	0.01	0.001	0.0001
$\frac{e^x-1}{x}$	1.052	1.005	1.0005	1.00005

Based on the numerical information provided in the table above, we conclude that

$$\lim_{x\to 0^+} \frac{e^x - 1}{x} = 1.$$

Again, observe that plugging $x = 0$ into the function $f(x) = \dfrac{e^x - 1}{x}$ yields the indeterminate form $\frac{0}{0}$.

NL 4. Compute the following limit:

$$\lim_{x\to 0} \frac{\sin(x)}{x}.$$

This is a two-sided limit. We begin be computing each of the corresponding one-sided limits.

To compute the right side limit, we consider the following values for x: $.1, .01, .001$ and $.0001$. Plugging these values into the function we generate the following table of values:

x	0.1	0.01	0.001	0.0001
$\frac{\sin(x)}{x}$	0.98	0.998	0.9998	0.99998

Based on this numerical evidence, it would be reasonable to guess that

$$\lim_{x \to 0^+} \frac{\sin(x)}{x} = 1.$$

To compute the left side limit, we can consider the following values for x: $-0.1, -0.01, -0.001$ and -0.0001.

However, we can also make the following observation about the sine function:

$$\sin(-x) = -\sin(x),$$

which is to say that the sine function is **odd**.

As a result, we have the following

$$\frac{\sin(-x)}{-x} = \frac{-\sin(x)}{-x} = \frac{\sin(x)}{x},$$

which means that the suggested negative values for x in the left side limit will produce exactly the same values as their positive counterparts that were used in the right side limit.

Thus,

$$\lim_{x \to 0^-} \frac{\sin(x)}{x} = \lim_{x \to 0^+} \frac{\sin(x)}{x} = 1.$$

Moreover, since both one sided limits are the same, we can dispense with the direction and conclude

$$\lim_{x \to 0} \frac{\sin(x)}{x} = 1.$$

Note, once again that the shortcut of plugging $x = 0$ into the function $f(x) = \dfrac{\sin(x)}{x}$ yields the indeterminate form $\frac{0}{0}$.

NL 5. Consider the limit:

$$\lim_{x\to -2^-} \frac{3}{x+2}.$$

First we observe that plugging in the value $x = -2$ gives $\frac{3}{0}$ which is undefined. Thus we are required to make a deeper analysis to determine the limit. Since the values of x are less than -2, we consider values for x such as $-2.1, -2.01, -2.001$ and -2.0001 to make the following table of values:

x	-2.1	-2.01	-2.001	-2.0001
$\frac{3}{x+2}$	-30	-300	-3,000	-30,000

The numerical evidence suggests that as x approaches -2 from the left, the values of $f(x) = \frac{2}{x-3}$ are decreasing without bound. We conclude that

$$\lim_{x\to -2^-} \frac{3}{x+2} = -\infty.$$

This result has geometric significance. It means that the line $x = -2$ is a vertical asymptote for the graph of the function $f(x) = \frac{3}{x+2}$.

NL 6. Consider the limit:

$$\lim_{x\to\frac{\pi}{2}^-} \tan(x).$$

Since

$$\tan(x) = \frac{\sin(x)}{\cos(x)},$$

$$\sin\left(\frac{\pi}{2}\right) = 1 \text{ and } \cos\left(\frac{\pi}{2}\right) = 0$$

we have

$$\tan\left(\frac{\pi}{2}\right) = \frac{1}{0}$$

which is undefined, so a deeper analysis is required to determine the limit. To understand the behavior of the tangent function for values of x near $\frac{\pi}{2}$, use a calculator to make a table of values noting that $\frac{\pi}{2} \approx 1.570796327$.

x	1.5	1.57	1.5707	1.57079
$\tan(x)$	14.101	1,255.77	10,381.33	158,057.91

The numerical evidence suggests that as x approaches $\frac{\pi}{2}$ from the left, the values of $f(x) = \tan(x)$ are increasing without bound. Therefore, we are led to conclude that

$$\lim_{x\to\frac{\pi}{2}^-} \tan(x) = \infty.$$

This result has geometric significance. It means that the graph of the function $f(x) = \tan(x)$ has a vertical asymptote at $x = \frac{\pi}{2}$.

NL 7. Consider the limit:

$$\lim_{x \to -\infty} \frac{2x+1}{3x-4}.$$

Since $x \to -\infty$, we will use the following values when constructing our table: $-10, -100, -1,000$ and $-10,000$.

x	-10	-100	-1000	-10000
$\frac{2x+1}{3x-4}$	.5588	.6546	.6654	.6665

The numerical evidence suggests that as x approaches $-\infty$, that is, as x decreases without bound, the values of $\frac{2x+1}{3x-4}$ are approaching the decimal .6666... which we recognize as the fraction 2/3. Hence,

$$\lim_{x \to -\infty} \frac{2x+1}{3x-4} = \frac{2}{3}.$$

This result has geometric significance. It means that the line $y = 2/3$ is a horizontal asymptote for the graph of the function $f(x) = \frac{2x+1}{3x-4}$.