Fall 2015 Math 344 Test III Name: Solutions

Show all work and provide brief explanations where necessary.

1. Find the eigenvalues of the matrix A and show that A is not diagonalizable.

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\det(\lambda I - A) = \det\begin{pmatrix} \lambda-1 & -2 & 0 \\ 0 & \lambda-1 & 0 \\ -1 & 0 & \lambda \end{pmatrix}$$

$$= \underbrace{\lambda(\lambda-1)^2}_{\text{char. poly}} = 0 \Rightarrow \underbrace{\lambda = 0}_{\text{mult.}=1} \text{ or } \underbrace{\lambda = 1}_{\text{mult}=2}$$

If $\dim E_1 \neq 2$ then A is not diagonalizable

$E_1 = \{\vec{x} \mid A\vec{x} = \vec{x}\}$

Solve the system $\left[\begin{array}{ccc|c} 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \end{array}\right]$

$$\xrightarrow{RREF} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\Rightarrow x_3 = t, \quad x_2 = 0, \quad x_1 = t$

$$\vec{x} = \begin{bmatrix} t \\ 0 \\ t \end{bmatrix} = t\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$\Rightarrow E_1 = \text{Span}\left\{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right\} \Rightarrow \dim E_1 = 1 \neq 2$

$\Rightarrow A$ is not diagonalizable.

2. Show that if λ is an eigenvalue for a matrix A with associated eigenvector $\mathbf{x}$, then λ^2 is an eigenvalue for A^2 with associated eigenvector $\mathbf{x}$.

Given $A\vec{x} = \lambda\vec{x}$

we have $A^2\vec{x} = A(A\vec{x}) = A(\lambda\vec{x})$

$= \lambda(A\vec{x}) = \lambda(\lambda\vec{x}) = \lambda^2\vec{x}$

So λ^2 is an eigenvalue of A^2
w/ eigenvector $\vec{x}$.

3. Find the eigenvalues of the matrix A and diagonalize A, i.e. find a diagonal matrix D and a non-singular matrix P such that $A = PDP^{-1}$. $A = \begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix}$

$\det(\lambda I - A) = \det\begin{pmatrix} \lambda-1 & 2 \\ 2 & \lambda-4 \end{pmatrix} = (\lambda-1)(\lambda-4) - 4$

$= \lambda^2 - 5\lambda + 4 - 4 = \lambda^2 - 5\lambda = \lambda(\lambda-5) = 0$

$\Rightarrow \lambda = 0, 5$ eigenvalues

$\lambda = 0$, E_0: $\left[\begin{array}{cc|c} -1 & 2 & 0 \\ 2 & -4 & 0 \end{array}\right] \to \left(\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right)$ $\begin{array}{l} x_2 = t \\ x_1 = 2t \end{array}$ $\vec{x} = t\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$\lambda = 5$, E_5: $\begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} \to \left[\begin{array}{cc|c} 2 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$ $\begin{array}{l} x_2 = t \\ x_1 = -\frac{1}{2}t \end{array}$ $\vec{x} = t\begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$ or $t\begin{bmatrix} -1 \\ 2 \end{bmatrix}$

So $A = PDP^{-1}$ with $D = \begin{bmatrix} 0 & 0 \\ 0 & 5 \end{bmatrix}$ and $P = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$

4. Orthogonally diagonalize the symmetric matrix A. That is, find a diagonal matrix D and an orthogonal matrix P such that $A = PDP^T$. $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

$$\det(\lambda I - A) \doteq \det\begin{pmatrix} \lambda & -1 & -1 \\ -1 & \lambda & -1 \\ -1 & -1 & \lambda \end{pmatrix} \leftarrow \text{laplace, row 1}$$

$$= \lambda(\lambda^2 - 1) + (-\lambda - 1) - (1+\lambda)$$

$$= \lambda(\lambda+1)(\lambda-1) - (\lambda+1) - (\lambda+1)$$

$$= (\lambda+1)[\lambda(\lambda-1) - 1 - 1]$$

$$= \lambda+1\,[\lambda^2 - \lambda - 2] = (\lambda+1)(\lambda+1)(\lambda-2) = 0$$

$\lambda = -1, 2$ eigenvalues $\Rightarrow$ $\boxed{D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}}$

Eigenvectors $\lambda = -1$:

$$\begin{pmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \begin{aligned} x_1 &= -s-t \\ x_2 &= s \\ x_3 &= t \end{aligned}$$

eigenvectors $s\underbrace{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}}_{v_1} + t\underbrace{\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}}_{v_2}$ $\Rightarrow$ Gram-Schmidt

$$w_2 = v_2 - \text{proj}_{v_1} v_2$$

$$\lambda = 2: \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & -2 \\ 0 & 3 & -3 \\ 0 & -3 & 3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - \left(\frac{1}{2}\right)\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix} \sim \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$$

$x_1 = t$
$x_2 = t$
$x_3 = t$

$$\vec{x} = t\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\boxed{P = \begin{bmatrix} \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}}$$

5. Are the following transformations from P_2 to P_1 <u>linear</u> transformations? Justify your answers.

a) $L(at^2 + bt + c) = at + c^2$

$$L(r(at^2+bt+c)) = L(rat^2 + rbt + rc)$$
$$= rat + (rc)^2$$

and $r\,L(at^2+bt+c) = r(at+c^2) = rat + rc^2$

but if $r \neq 0, 1$ then

$(rc)^2 \neq rc^2$

So L is <u>not</u> linear

b) $L(at^2 + bt + c) = 2at + b$ ← the derivative is linear.

(1) $L((at^2+bt+c) + (a't^2+b't+c'))$
$$= L((a+a')t^2 + (b+b') + (c+c'))$$
$$= 2(a+a') + (b+b')$$
$$= (2a+b) + (2a'+b') = L(at^2+bt+c) + L(a't^2+b't+c')$$

(2) $L(r(at^2+bt+c)) = L(rat^2 + rbt + rc)$
$$= 2ra + rb = r(2a+b)$$
$$= r\,L(at^2+bt+c).$$

L <u>is</u> linear.

6. Suppose $L : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation such that $L(1,1) = (3,2)$ and $L(1,-1) = (1,4)$. Find $L(3,1)$. Hint: express $(3,1)$ in terms of $(1,1)$ and $(1,-1)$.

$$(3,1) = 2(1,1) + (1,-1)$$

by solving $(3,1) = a(1,1) + b(1,-1)$ for a, b.

Thus $L(3,1) = 2L(1,1) + L(1,-1)$

(since L is linear)

$$= 2(3,2) + (1,4)$$

$$= (6,4) + (1,4)$$

$$= (7,8)$$

7. Find Ker(L) if $L : \mathbb{R}^3 \to \mathbb{R}^2$ is defined by $L(x, y, z) = (x + z, y - z)$. Is L a one-to-one transformation? Explain.

$$\text{Ker}(L) = \{ (x,y,z) \mid L(x,y,z) = (0,0) \}$$

Solve $\begin{matrix} x + z = 0 \\ y - z = 0 \end{matrix} \to \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{array} \right)$ $z = t,\ y = t,\ x = -t$

$$\text{Ker}(L) = \{ (-t, t, t) \} = \text{span}\{ (-1, 1, 1) \}.$$

L is not 1-1 since $\text{Ker}(L) \neq \{ \vec{0} \}$.

8. Find the matrix of the transformation $L : \mathbb{R}^2 \to \mathbb{R}^3$ by $L(x, y) = (x+y, 2y-x, 3x-2y)$.

A is a 3×2 matrix.

$$A = \left[\underset{\text{col } 1}{L(1,0)} \quad \underset{\text{col } 2}{L(0,1)} \right] = \begin{bmatrix} 1 & 1 \\ -1 & 2 \\ 3 & -2 \end{bmatrix}$$ since

$L(1,0) = (1,-1,3)$ and

$L(0,1) = (1, 2, -2)$.