Fall 2015 Math 344 Test II Name: Solutions

Show all work and provide brief explanations where necessary.

1. Let V be the set $\mathbb{R}^3$ and define the operations of addition and scalar multiplication by $(x, y, z) \oplus (x', y', z') = (x + x', y + y', z + z')$ and $c \odot (x, y, z) = (cx, cy, z)$ respectively. Show that with these operations, V is not a vector space.

Property (f) fails.

(f) says $(c+d) \odot u = (c \odot u) \oplus (d \odot u)$

LHS: $(c+d) \odot u = (c+d) \odot (x,y,z)$
$= ((c+d)x, (c+d)y, z)$.

RHS: $(c \odot u) \oplus (d \odot u) = (c \odot (x,y,z)) \oplus (d \odot (x,y,z))$
$= (cx, cy, z) \oplus (dx, dy, z)$
$= (cx+dx, cy+dy, 2z)$

Now $((c+d)x, (c+d)y, z) \neq (cx+dx, cy+dy, 2z)$

if $z \neq 0$.

2. Which of the following subsets of P_2 are subspaces? Justify your answers.

a) $S = \{at^2 + bt + c \mid a + b + c \geq 0\}$

Not a subspace

(β) fails if the scalar is negative

let $u = t^2 \in S$

$r = -3$

$r \odot u = -3t^2 \notin S$

S is not closed under $\odot$

b) $T = \{at^2 + bt + c \mid a + b + c = 0\}$ T is a subspace of P_2

Let $v_1 = a_1 t^2 + b_1 t + c_1$ with $a_1 + b_1 + c_1 = 0$

and $v_2 = a_2 t^2 + b_2 t + c_2$ with $a_2 + b_2 + c_2 = 0$

then $v_1 + v_2 = (a_1 + a_2)t^2 + (b_1 + b_2)t + (c_1 + c_2)$

and $(a_1 + a_2) + (b_1 + b_2) + (c_1 + c_2) = (a_1 + b_1 + c_1) + (a_2 + b_2 + c_2)$

$= 0 + 0 = 0$, so $v_1 + v_2 \in T$

T is closed under $\oplus$

(β) $\underset{\in \mathbb{R}}{r} \odot \underset{\in T}{v} = r \odot (at^2 + bt + c) = rat^2 + rbt + rc$

and $ra + rb + rc = r(a+b+c) = r \cdot 0 = 0$

T is closed under $\odot$

3. Is the set $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \right\}$ linearly independent in M_{22}? Justify your answer.

Suppose $c_1 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

then

$$\begin{aligned} c_1 + c_2 \quad &= 0 \\ c_2 + c_3 &= 0 \\ c_2 + c_3 &= 0 \\ c_1 \quad + c_3 &= 0 \end{aligned} \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{R_4 - R_1} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix} \xrightarrow[R_4 + R_2]{R_3 - R_2} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right]$$

$$\xrightarrow[R_3 \leftrightarrow R_4]{\frac{1}{2} R_4} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

so $c_3 = 0$ (from row 3)

$c_2 + c_3 = 0 \Rightarrow c_2 = 0$ (row 2)

$c_1 + c_2 = 0 \Rightarrow c_1 = 0$ (row 1)

Since the only solution is the trivial solution,

$c_1 = 0,\ c_2 = 0,\ c_3 = 0,$

the vectors are linearly independent

4. Determine whether or not each of the following sets of vectors is a basis of $\mathbb{R}^3$. Justify your answers.

a) $\{(1,0,1),(0,1,1)\}$

dim $(\mathbb{R}^3) = 3$ so a basis has 3 vectors.

This set only has 2 vectors, so it is not a basis.

b) $\{(1,0,1),(0,1,1),(1,1,1),(0,0,1)\}$

This set has 4 vectors and $4 \neq 3$, so this set is not a basis.

c) $\{(1,0,1),(0,1,1),(1,1,1)\}$ ← 3 vectors. Are they lin. indep?

$$c_1(1,0,1) + c_2(0,1,1) + c_3(1,1,1) = (0,0,0)$$

$$\Rightarrow \quad c_1 + c_3 = 0, \quad c_2 + c_3 = 0, \quad c_1 + c_2 + c_3 = 0$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right) \xrightarrow{R_3 - R_1} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{R_3 - R_2} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right) \Rightarrow c_3 = 0, \; c_2 = 0, \; c_1 = 0.$$

$\Rightarrow$ lin. indep. $\Rightarrow$ basis

5. Find a basis for the null space of the matrix $A = \begin{bmatrix} 1 & 3 & 0 & 1 \\ 0 & 1 & 2 & 3 \\ 1 & 5 & 4 & 7 \end{bmatrix}$.

$$\xrightarrow{R_3 - R_1} \begin{bmatrix} 1 & 3 & 0 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 2 & 4 & 6 \end{bmatrix} \xrightarrow{R_3 - 2R_1} \begin{bmatrix} 1 & 3 & 0 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R_1 - 3R_2} \begin{bmatrix} 1 & 0 & -6 & -8 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The system $A\vec{x} = 0$ with $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$

leads to $\left[\begin{array}{cccc|c} 1 & 0 & -6 & -8 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

x_3 and x_4 are free variables, so $x_3 = s$, $x_4 = t$.

and $x_1 = 6s + 8t$, $x_2 = -2s - 3t$.

Thus $\vec{x} = \begin{bmatrix} 6s+8t \\ -2s-3t \\ s \\ t \end{bmatrix} = \begin{bmatrix} 6s \\ -2s \\ s \\ 0 \end{bmatrix} + \begin{bmatrix} 8t \\ -3t \\ 0 \\ t \end{bmatrix}$

$= s\begin{bmatrix} 6 \\ -2 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 8 \\ -3 \\ 0 \\ 1 \end{bmatrix}$

$$\text{Null}(A) = \text{span}\underbrace{\left\{ \begin{bmatrix} 6 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 8 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right\}}_{\text{basis}}$$

6. Find bases for the column space and the row space of the matrix $A = \begin{bmatrix} 1 & 3 & 2 & 5 \\ 1 & 3 & 3 & 6 \\ 1 & 3 & 1 & 4 \\ 0 & 0 & 2 & 2 \end{bmatrix}$.

$$\xrightarrow[R_3-R_1]{R_2-R_1} \begin{bmatrix} 1 & 3 & 2 & 5 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 2 & 2 \end{bmatrix} \xrightarrow[R_4-2R_2]{R_3+R_2} \begin{bmatrix} 1 & 3 & 2 & 5 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R_1-2R_2} \begin{bmatrix} 1 & 3 & 0 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\text{Row}(A) = \text{span}\underbrace{\{(1,3,0,3), (0,0,1,1)\}}_{\text{basis}}$$

$$\text{Col}(A) = \text{span}\{\text{col } 1, \text{col } 3 \text{ of } A\}$$

$$= \text{Span}\underbrace{\left\{\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 1 \\ 2 \end{bmatrix}\right\}}_{\text{basis.}}$$

7. Use the Gram-Schmidt Process to find an orthonormal basis of the subspace W of $\mathbb{R}^4$ given by $W = \text{span}\{(1,0,1,0),(1,0,1,1),(0,1,1,1)\}$.

$W_1 \quad W_2 \quad W_3$

let $v_1 = w_1 = \boxed{(1,0,1,0) = v_1}$

let $v_2 = w_2 - \text{proj}_{v_1}(w_2) = w_2 - \left(\frac{v_1 \cdot w_2}{v_1 \cdot v_1}\right) v_1$

$= (1,0,1,1) - \left(\frac{2}{2}\right)(1,0,1,0) = \boxed{(0,0,0,1) = v_2}$

let $v_3 = w_3 - \text{proj}_{v_1}(w_3) - \text{proj}_{v_2}(w_3)$

$= w_3 - \left(\frac{v_1 \cdot w_3}{v_1 \cdot v_1}\right) v_1 - \left(\frac{v_2 \cdot w_3}{v_2 \cdot v_2}\right) v_2$

$= (0,1,1,1) - \frac{1}{2}(1,0,1,0) - \frac{1}{1}(0,0,0,1)$

$= \left(-\frac{1}{2}, 1, \frac{1}{2}, 0\right)$; New $\boxed{v_3 = (-1, 2, 1, 0)}$

Normalize :

$u_1 = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}, 0\right)$, $u_2 = (0,0,0,1)$

$u_3 = \left(\frac{-1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, 0\right)$

O.N. basis for W : $\{u_1, u_2, u_3\}$.

8. Let $\mathbf{v}_1, \mathbf{v}_2$ and $\mathbf{v}_3$ be vectors in a vector space such that $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly independent. Show that if $\mathbf{v}_3$ does not belong to span$\{\mathbf{v}_1, \mathbf{v}_2\}$, then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent.

Consider: $c_1 v_1 + c_2 v_2 + c_3 v_3 = 0$

suppose $c_3 \neq 0$ then

$$v_3 = -\frac{c_1}{c_3} v_1 - \frac{c_2}{c_3} v_2$$

$\Rightarrow$ $v_3 \in$ span $\{v_1, v_2\}$ $\longrightarrow \longleftarrow$ (contradiction)

$\therefore$ $c_3 = 0$.

This leaves $c_1 v_1 + c_2 v_2 + 0 v_3 = \vec{0}$

$\Rightarrow$ $c_1 v_1 + c_2 v_2 = \vec{0}$

$\Rightarrow$ $c_1 = 0$ and $c_2 = 0$

since $\{v_1, v_2\}$ lin. indep.

thus $c_1 = 0$, $c_2 = 0$ and $c_3 = 0$

$\Rightarrow$ $\{v_1, v_2, v_3\}$ lin. indep.