Fall 2015 Math 344 Test I Name: Solutions

Show all work and provide brief explanations where necessary.

1. Solve the linear system by reducing the augmented matrix to row echelon (or reduced row echelon) form.

$$\begin{array}{rcrcrcr} x & + & y & + & z & = & 4 \\ 2x & - & y & + & 3z & = & 7 \\ 4x & + & y & + & 5z & = & 15 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 2 & -1 & 3 & 7 \\ 4 & 1 & 5 & 15 \end{array}\right] \xrightarrow[R_3-4R_1]{R_2-2R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & -3 & 1 & -1 \\ 0 & -3 & 1 & -1 \end{array}\right]$$

$$\xrightarrow{R_3-R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & -3 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{-\frac{1}{3}R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 1 & -\frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 0 \end{array}\right]$$

REF

Solution: $\boxed{z = t}$, $\boxed{y = \frac{1}{3} + \frac{1}{3}t}$. and $x+y+z=4$

$\Rightarrow x = 4 - y - z$

$= 4 - \frac{1}{3} - \frac{1}{3}t - t$

$\boxed{x = \frac{11}{3} - \frac{4}{3}t}$

2. Let $A = \begin{bmatrix} 1 & 3 & 0 \\ -1 & 1 & 5 \end{bmatrix}$, and $B = \begin{bmatrix} 1 & 3 \\ -1 & 1 \\ 0 & 1 \end{bmatrix}$. Find $2A - 3B^T$.

$2A = \begin{bmatrix} 2 & 6 & 0 \\ -2 & 2 & 10 \end{bmatrix}$, $B^T = \begin{bmatrix} 1 & -1 & 0 \\ 3 & 1 & 1 \end{bmatrix}$, $3B^T = \begin{bmatrix} 3 & -3 & 0 \\ 9 & 3 & 3 \end{bmatrix}$.

and $2A - 3B^T = \begin{bmatrix} -1 & 9 & 0 \\ -11 & -1 & 7 \end{bmatrix}$

3. For the matrices in problem 2, find AB. Also, explain why $BA \neq AB$ without computing BA.

$$A = \begin{bmatrix} 1 & 3 & 0 \\ -1 & 1 & 5 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 3 \\ -1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$AB = \begin{bmatrix} (1)(1) + (3)(-1) + (0)(0) & (1)(3) + (3)(1) + (0)(1) \\ (-1)(1) + (1)(-1) + (5)(0) & (-1)(3) + (1)(1) + (5)(1) \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 6 \\ -2 & 3 \end{bmatrix}.$$

BA is a 3×3 mtx, so $BA \neq AB$

4. Determine a scalar, r, such that $A\mathbf{x} = r\mathbf{x}$ where $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, and $\mathbf{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

$$A\mathbf{x} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

so $\boxed{r = 3}$

5. Find A if $A^{-1} = \begin{bmatrix} 0 & 1 & -2 \\ 1 & 0 & 3 \\ 2 & 0 & 8 \end{bmatrix}$.

$A = (A^{-1})^{-1}$

$$\left[\begin{array}{ccc|ccc} 0 & 1 & -2 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 2 & 0 & 8 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 \\ 2 & 0 & 8 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_3 - 2R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & -2 & 1 \end{array}\right] \xrightarrow{\frac{1}{2}R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & \frac{1}{2} \end{array}\right]$$

$$\xrightarrow[R_2 + 2R_3]{R_1 - 3R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 4 & -3/2 \\ 0 & 1 & 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & 0 & -1 & \frac{1}{2} \end{array}\right] \qquad A = \begin{bmatrix} 0 & 4 & -3/2 \\ 1 & -2 & 1 \\ 0 & -1 & \frac{1}{2} \end{bmatrix}$$

6. Use a Laplace cofactor expansion to find $\det(A)$ if $A = \begin{bmatrix} 3 & -1 & -2 \\ 1 & 4 & -3 \\ 2 & 3 & 0 \end{bmatrix}$.

row 3: $2(-1)^{3+1} \det\begin{pmatrix} -1 & -2 \\ 4 & -3 \end{pmatrix} + 3(-1)^{3+2} \det\begin{pmatrix} 3 & -2 \\ 1 & -3 \end{pmatrix}$

$= 2(3+8) + 3(-1)(-9+2)$

$= 22 + (-3)(-7)$

$= 43.$

7. If A is a 2 x 2 matrix with $\det(A)$ = -3, find $\det(A^{-1})$, $\det(A^T)$,$\det(4A)$ and $\det(A^2)$.

$\det(A^{-1}) = \frac{1}{\det(A)} = -\frac{1}{3}$; $\det(A^T) = \det(A) = -3$

$\det(4A) = 4^2 \cdot \det(A) = 16(-3) = -48$

$\det(A^2) = \det(A \cdot A) = \det(A) \cdot \det(A) = (-3)(-3) = 9.$

8. Use determinants to determine whether each of the following matrices is singular or nonsingular:

$$A = \begin{bmatrix} 3 & -2 \\ 6 & -4 \end{bmatrix}, \quad B = \begin{bmatrix} -3 & -2 \\ -6 & 4 \end{bmatrix}, \quad C = \begin{bmatrix} 3 & -1 \\ -12 & 4 \end{bmatrix}.$$

$\det(A) = -12 - (-12) = 0$ so A is singular.

$\det(B) = -12 - 12 = -24 \neq 0$ so B is non-singular

$\det(C) = 12 - 12 = 0$ so C is singular

9. Find the angle between the vectors $\mathbf{u} = (1, 2, -3, -4)$ and $\mathbf{v} = (1, 0, -2, 5)$.

$$\cos\theta = \frac{u \cdot v}{\|u\| \cdot \|v\|} = \frac{(1)(1) + (2)(0) + (-3)(-2) + (-4)(5)}{\sqrt{1^2+2^2+3^2+4^2} \cdot \sqrt{1^2+0^2+2^2+5^2}}$$

$$= \frac{-13}{\sqrt{30}\sqrt{30}} = -\frac{13}{30}$$

$$\theta = \cos^{-1}\left(-\frac{13}{30}\right) \approx 115.7^\circ$$

10. Let $\mathbf{u}$ and $\mathbf{v}$ be solutions of the homogeneous linear system $A\mathbf{x} = \mathbf{0}$. Show that $3\mathbf{u} - 5\mathbf{v}$ is also a solution.

Given $Au = \vec{0}$ and $Av = \vec{0}$.

$$A(3u - 5v) = A(3u) - A(5v) = 3Au - 5Av$$

$$= 3\vec{0} - 5\vec{0} = \vec{0},$$

so $3u - 5v$ is also a solution.

11. Show that if $\det(A^T) = \det(A^{-1})$ then $\det(A) = \pm 1$.

Relevant facts: $\det(A^T) = \det(A)$ and $\det(A^{-1}) = \frac{1}{\det(A)}$.

Since $\det(A^T) = \det(A^{-1})$, we have

$$\det(A) = \frac{1}{\det(A)}$$

$$\Rightarrow \det(A)^2 = 1$$

$$\Rightarrow \det(A) = \pm 1.$$

12. Show that if $\mathbf{u} \cdot \mathbf{v} = 0$ for all n-vectors, $\mathbf{v}$, then $\mathbf{u} = \mathbf{0}$ (the zero vector).

let $v = u$

$$u \cdot u = 0 \Rightarrow \|u\|^2 = 0$$

$$\Rightarrow \|u\| = 0$$

$$\Rightarrow u = \vec{0}.$$