Homework 13 solutions

p. 507

3b) $L(0) = 1 \neq \vec{0}$ so L is not linear

3c) $L(at+b) = at^2 + (a-b)t$ is linear b/c

$$L((at+b) + (a't+b')) = L((a+a')t + (b+b'))$$

$$= (a+a')t^2 + [(a+a') - (b+b')]t$$

$$= [at^2 + (a-b)t] + [a't^2 + (a'-b')t]$$

$$= L(at+b) + L(a't+b')$$

and $L(r(at+b)) = L(rat + rb) = rat^2 + (ra-rb)t$

$$= r[at^2 + (a-b)t]$$
$$= rL(at+b).$$

10) $L\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = a+b-c-d+1$ is not linear b/c

$L\left(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right) = 1 \neq 0.$

p507 (continued)

18) $L(1) = 1$, $L(t) = t^2$, $L(t^2) = t^3 + t$

a) $L(2t^2 - 5t + 3) = 2L(t^2) - 5L(t) + 3L(1)$
since L is linear.

$= 2[t^3 + t] - 5[t^2] + 3[1]$

$= 2t^3 + 2t - 5t^2 + 3 = 2t^3 - 5t^2 + 2t + 3.$

b) $L(at^2 + bt + c) = aL(t^2) + bL(t) + cL(1)$

$= a[t^3 + t] + b[t^2] + c[1]$

$= at^3 + bt^2 + at + c$

T11) $L^{-1}(W_1) = \{v \in V \mid L(v) \in W_1\}$ when W_1 is a subspace of W

Let $v_1, v_2 \in L^{-1}(W_1)$ then $L(v_1 + v_2) = L(v_1) + L(v_2)$

$= w_1 + w_2$ where $w_1 \& w_2 \in W_1$.

Since W_1 is a subspace of W, $w_1 + w_2 \in W_1$ (closed under +)

$\Rightarrow v_1 + v_2 \in L^{-1}(W_1)$ i.e. $L^{-1}(W_1)$ is closed under +

p507 (continued)

T11) (continued)

If $r \in \mathbb{R}$ and $v_1 \in L^{-1}(W_1)$ then

$$L(rv_1) = rL(v_1) = rw_1 \text{ where } w_1 \in W_1$$

Since W_1 is a subspace of W, $rw_1 \in W_1$ (closed under $\cdot$)

thus $rv_1 \in L^{-1}(W_1)$. i.e. $L^{-1}(W_1)$ is closed under $\cdot$

Since $L^{-1}(W_1)$ is closed under both addition and scalar multiplication, it is a subspace of V.

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2) $L\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$

a) $L\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ so $\begin{bmatrix} 1 \\ 2 \end{bmatrix} \notin \text{Ker}(L)$

c) $L\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix} \Rightarrow a_1 = 1,\ a_2 = 1$ (works)

$\Rightarrow L\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix} \Rightarrow \begin{bmatrix} 3 \\ 6 \end{bmatrix} \in \text{Range } L$.

e) Ker(L): $L\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\left(\begin{array}{cc|c} 1 & 2 & 0 \\ 2 & 4 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right)$ $\begin{matrix} a_1 = -2t \\ a_2 = t \end{matrix}$ $\Rightarrow \text{Ker}(L) = \left\{t\begin{bmatrix} -2 \\ 1 \end{bmatrix}\right\}$

$= \text{span}\left\{\begin{bmatrix} -2 \\ 1 \end{bmatrix}\right\}$

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4) $L(x,y,z,w) = (x+y,\ z+w,\ x+z)$ can be written as

$$L\left(\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}\right) = A\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$ for a 3×4 mtx A

$$A = \begin{bmatrix} L(e_1) & L(e_2) & L(e_3) & L(e_4) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

$$\left(e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix},\ e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix},\ e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix},\ e_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right)$$

a) $\operatorname{Ker}(L) = \operatorname{Null}(A)$ so find RREF of A:

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 \end{array}\right]$$

$$\to \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right]$$

$\Rightarrow$ $x = t$, $y = -t$, $z = -t$, $w = t$ $\Rightarrow$ $\operatorname{Ker}(L) = \operatorname{span}\left\{\begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}\right\}$

free variable $\longrightarrow$ $w = t$

col's 1,2,3 of A

b) Range $L = \operatorname{Col}(A) = \operatorname{Span}\left\{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right\}$ (verify these 3 vectors are lin. indep.)

c) $\dim \operatorname{Ker}(L) = 1$, $\dim \operatorname{Range} L = 3$ and $1+3 = 4 = \dim V$.

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6) $L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \overset{A}{\begin{bmatrix} 4 & 2 & 2 \\ 2 & 3 & -1 \\ -1 & 1 & -2 \end{bmatrix}} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

a) $\ker(L) = \text{null}(A)$

$$\left[\begin{array}{ccc|c} 4 & 2 & 2 & 0 \\ 2 & 3 & -1 & 0 \\ -1 & 1 & -2 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ 0 & 5 & -5 & 0 \\ 0 & 6 & -6 & 0 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad z = t, \; y = t, \; x = -t$$

$$\ker(L) = \text{span}\left\{\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}\right\} \neq \{\vec{0}\}$$

since $\ker(L) \neq \{\vec{0}\}$, L is <u>not</u> one-to-one

7) $\dim(\ker L) + \dim(\text{Range } L) = \dim(\text{domain } L)$

$\Rightarrow \quad 1 + \dim(\text{Range } L) = 3$

$\Rightarrow \quad \dim(\text{Range } L) = 2$

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11) $L\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{bmatrix} 1 & 3 & 1 \\ 1 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ so

a) $L\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 1 \\ 1 & 2 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 10 \\ 5 \\ 5 \end{bmatrix}$

b) $L\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{bmatrix} 1 & 3 & 1 \\ 1 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 2 \end{bmatrix}$

16) $L(e_1) = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $L(e_2) = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$, $L(e_3) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

so $L\begin{pmatrix} x \\ y \\ z \end{pmatrix} = A\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ when

a) $A = \begin{bmatrix} L(e_1) & L(e_2) & L(e_3) \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$

b) $L\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = A\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 8 \\ 1 \\ 5 \end{bmatrix}$

c) $L\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a+2b+c \\ b \\ b+c \end{bmatrix}$