Linear Algebra HW #8 Answer Key

$\boxed{6.}$

$$A = \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 1 & 2 & 3 & 1 & 0 \\ 2 & 1 & 3 & 1 & 0 \\ 1 & 1 & 2 & 1 & 0 \end{array}\right] \xrightarrow[R_4 - R_1]{\substack{R_2 - R_1 \\ R_3 - 2R_1}} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 2 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{\frac{1}{2}R_2} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1/2 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{array}\right] \xrightarrow[R_4 - R_2]{R_3 - R_2} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1/2 & 0 \\ 0 & 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1/2 & 0 \end{array}\right]$$

$$\xrightarrow{\substack{2R_3 \\ 2R_4}} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1/2 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow{R_4 - R_3} \begin{array}{c} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1/2 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \\ (x_1)(x_2)(x_3)(x_4) \end{array}$$

Let $x_3 = t, \ x_4 = 0$

$$x_2 + x_3 + \tfrac{1}{2}x_4 = x_2 + t + \tfrac{1}{2}(0) = 0$$
$$x_2 = -t$$

$$x_1 + x_3 = 0$$
$$x_1 + t = 0$$
$$x_1 = -t$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -t \\ -t \\ t \\ 0 \end{bmatrix} = t\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}$$

$$\boxed{\text{Solution: } \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}}$$

10b. $\{(4,2,1), (2,6,-5), (1,-2,3)\}$

$$\left[\begin{array}{ccc|c} 4 & 2 & 1 & 0 \\ 2 & 6 & -2 & 0 \\ 1 & -5 & 3 & 0 \end{array}\right] \xleftrightarrow{R_3 \quad R_1} \left[\begin{array}{ccc|c} 1 & -5 & 3 & 0 \\ 2 & 6 & -2 & 0 \\ 4 & 2 & 1 & 0 \end{array}\right]$$

$$\xrightarrow[R_3-4R_1]{R_2-2R_1} \left[\begin{array}{ccc|c} 1 & -5 & 3 & 0 \\ 0 & 16 & -8 & 0 \\ 0 & 22 & -11 & 0 \end{array}\right] \xrightarrow[\frac{1}{22}R_3]{\frac{1}{16}R_2} \left[\begin{array}{ccc|c} 1 & -5 & 3 & 0 \\ 0 & 1 & -1/2 & 0 \\ 0 & 1 & -1/2 & 0 \end{array}\right]$$

$$\xrightarrow{R_3-R_2} \left[\begin{array}{ccc|c} 1 & -5 & 3 & 0 \\ 0 & 1 & -1/2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{array}{l} x_3 = t \\ x_2 = \frac{1}{2}t \\ x_1 = -\frac{1}{2}t \end{array} \Rightarrow t\begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

$$\Rightarrow -\frac{1}{2}(4,2,1) + \frac{1}{2}(2,6,-5) + (1,-2,3) = (0,0,0)$$

$$2\left(\frac{1}{2}(2,6,-5) + (1,-2,3)\right) = \left(\frac{1}{2}(4,2,1)\right)2$$

$$\boxed{(2,6,-5) + 2(1,-2,3) = (4,2,1)}$$

12c. $\{3t+1,\ 3t^2+1,\ 2t^2+t+1\}$

$(0t^2+3t+1),\ (3t^2+0t+1),\ (2t^2+t+1)$

$$\left[\begin{array}{ccc|c} 0 & 3 & 2 & 0 \\ 3 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right] \xRightarrow{R_3 \leftrightarrow R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 3 & 0 & 1 & 0 \\ 0 & 3 & 2 & 0 \end{array}\right]$$

$\xrightarrow{R_2-3R_1}$ $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -3 & -2 & 0 \\ 0 & 3 & 2 & 0 \end{array}\right]$ $\xrightarrow[\frac{1}{3}R_3]{-\frac{1}{3}R_2}$ $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 2/3 & 0 \\ 0 & 1 & 2/3 & 0 \end{array}\right]$

$\xrightarrow{R_3-R_2}$ $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 2/3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$x_3 = t$

$x_2 + \frac{2}{3}t = 0$

$x_2 = -\frac{2}{3}t$

$x_1 - \frac{2}{3}t + t = 0$

$x_1 = -\frac{1}{3}t$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t \begin{bmatrix} \frac{1}{3} \\ -\frac{2}{3} \\ 1 \end{bmatrix}$$

$-\frac{1}{3}(3t+1) - \frac{2}{3}(3t^2+1) + (2t^2+t+1) = 0$

$-\frac{1}{3}(3t+1) = \frac{2}{3}(3t^2+1) - (2t^2+t+1)$

$\boxed{(3t+1) = 3(2t^2+t+1) - 2(3t^2+1)}$

T5. $T = \{w_1, w_2, w_3\}$ where

$w_1 = v_1 + v_2, \quad w_2 = v_1 + v_3, \quad w_3 = v_2 + v_3$

$c_1(v_1+v_2) + c_2(v_1+v_3) + c_3(v_2+v_3) = 0$

$c_1 v_1 + c_1 v_2 + c_2 v_1 + c_2 v_3 + c_3 v_2 + c_3 v_3 = 0$

$v_1(c_1+c_2) + v_2(c_1+c_3) + v_3(c_2+c_3) = 0$

$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right]$ $\xrightarrow{R_2-R_1}$ $\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right]$

$\xrightarrow{R_3-R_2}$ $\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right]$ $\xrightarrow{\frac{1}{2}R_3}$ $\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$

$c_1 = 0$
$c_2 = 0 \Rightarrow$ Thus, T is linearly independent
$c_3 = 0$

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$\boxed{1.}$ a. and d. are bases of $\mathbb{R}^2$.

(a.) $\{(1,3), (1,-1)\}$

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 3 & -1 & 0 \end{array}\right] \xrightarrow{R_2 - 3R_1} \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & -4 & 0 \end{array}\right] \Rightarrow \frac{-1}{4} R_2 \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 1 & 0 \end{array}\right]$$

$\Rightarrow$ So, linearly independent

$$\left[\begin{array}{cc|c} 1 & 1 & a \\ 3 & -1 & b \end{array}\right] \xrightarrow{R_2 - 3R_1} \left[\begin{array}{cc|c} 1 & 1 & a \\ 0 & -4 & b-3a \end{array}\right] \xrightarrow{-\frac{1}{4}R_2} \left[\begin{array}{cc|c} 1 & 1 & a \\ 0 & 1 & \frac{3a-b}{4} \end{array}\right]$$

$\Rightarrow$ spans $\mathbb{R}^2$
$\Rightarrow$ Thus, $\{(1,3),(1,-1)\}$ is a basis of $\mathbb{R}^2$.

(d.) $\{(1,3), (-2,6)\}$

$$\left[\begin{array}{cc|c} 1 & -2 & 0 \\ 3 & 6 & 0 \end{array}\right] \xrightarrow{R_2 - 3R_1} \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 12 & 0 \end{array}\right] \xrightarrow{\frac{1}{12}R_2} \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 1 & 0 \end{array}\right]$$

$\Rightarrow$ So, linearly independent.

$$\left[\begin{array}{cc|c} 1 & -2 & a \\ 3 & 6 & b \end{array}\right] \xrightarrow{R_2 - 3R_1} \left[\begin{array}{cc|c} 1 & -2 & a \\ 0 & 12 & b-3a \end{array}\right] \xrightarrow{\frac{1}{12}R_2} \left[\begin{array}{cc|c} 1 & -2 & a \\ 0 & 1 & \frac{b-3a}{12} \end{array}\right]$$

$\Rightarrow$ spans $\mathbb{R}^2$
$\Rightarrow$ Thus, $\{(1,3),(1,-1)\}$ is a basis of $\mathbb{R}^2$.