

0.1 Linear Approximation

In this section we will use the tangent line to a function at a point to approximate the value of the function at a point close to the point of tangency. Given a function $y = f(x)$, the equation of the tangent line at the point where $x = x_0$ is given by

$$y - f(x_0) = f'(x_0)(x - x_0)$$

or

$$y = f(x_0) + f'(x_0)(x - x_0).$$

The main idea of this section is that if we let

$$L(x) = f(x_0) + f'(x_0)(x - x_0)$$

then

$$f(x_0) = L(x_0)$$

and

$$f(x) \approx L(x)$$

for values of x close to x_0 . The function $L(x)$ is called the linearization of $f(x)$ at $x = x_0$. The advantage of working with $L(x)$ is that values of a linear function are usually easy to compute. In a typical linear approximation problem, we are trying to approximate $f(x_1)$ and we need to choose x_0 and create $L(x)$. Once we have accomplished this, our solution is

$$f(x_1) \approx L(x_1).$$

There are two keys to choosing x_0 . First x_0 should be close to x_1 and second, we must be able to compute the exact value of $f(x_0)$.

Examples of Linear Approximation

LA1. We can approximate $\sqrt{18}$ using linear approximation as follows. Let $f(x) = \sqrt{x}$ and since 16 is near 18 and $\sqrt{16} = 4$, we let $x_0 = 16$. To create $L(x)$ we also need to compute $f'(16)$. Since

$$f'(x) = \frac{d}{dx}\sqrt{x} = \frac{d}{dx}(x^{1/2}) = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}},$$

we have

$$f'(16) = \frac{1}{2\sqrt{16}} = \frac{1}{8}.$$

Next,

$$\begin{aligned} L(x) &= f(x_0) + f'(x_0)(x - x_0) \\ &= f(16) + f'(16)(x - 16) \\ &= 4 + \frac{1}{8}(x - 16) \end{aligned}$$

and finally,

$$\begin{aligned} \sqrt{18} &\approx L(18) = 4 + \frac{1}{8}(18 - 16) \\ &= 4 + \frac{2}{8} = \frac{17}{4} = 4.25. \end{aligned}$$

LA2. We can approximate $\sqrt{99}$ using linear approximation as follows. Let $f(x) = \sqrt{x}$ and since 100 is near 99 and $\sqrt{100} = 10$, we let $x_0 = 100$. To create $L(x)$ we also need to compute $f'(100)$. Since

$$f'(x) = \frac{d}{dx}\sqrt{x} = \frac{d}{dx}(x^{1/2}) = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}},$$

we have

$$f'(100) = \frac{1}{2\sqrt{100}} = \frac{1}{20}.$$

Next,

$$\begin{aligned}L(x) &= f(x_0) + f'(x_0)(x - x_0) \\&= f(100) + f'(100)(x - 100) \\&= 10 + \frac{1}{20}(x - 100)\end{aligned}$$

and finally,

$$\begin{aligned}\sqrt{99} &\approx L(99) = 10 + \frac{1}{20}(99 - 100) \\&= 10 - \frac{1}{20} = \frac{199}{20} = 9.95.\end{aligned}$$

LA3. We can approximate $\sqrt[3]{10}$ using linear approximation as follows. Let $f(x) = \sqrt[3]{x}$ and since 10 is near 8 and $\sqrt[3]{8} = 2$, we let $x_0 = 8$. To create $L(x)$ we also need to compute $f'(8)$. Since

$$f'(x) = \frac{d}{dx}\sqrt[3]{x} = \frac{d}{dx}(x^{1/3}) = \frac{1}{3}x^{-2/3} = \frac{1}{3\sqrt[3]{x^2}},$$

we have

$$f'(8) = \frac{1}{3\sqrt[3]{8^2}} = \frac{1}{3\sqrt[3]{64}} = \frac{1}{12}.$$

Next,

$$\begin{aligned}L(x) &= f(x_0) + f'(x_0)(x - x_0) \\&= f(8) + f'(8)(x - 8) \\&= 2 + \frac{1}{12}(x - 8)\end{aligned}$$

and finally,

$$\begin{aligned}\sqrt[3]{10} &\approx L(10) = 2 + \frac{1}{12}(10 - 8) \\&= 2 + \frac{2}{12} = \frac{13}{6} = 2.1\bar{6}.\end{aligned}$$

LA4. We can approximate $e^{-0.1}$ using linear approximation as follows. Let $f(x) = e^x$ and since 0 is near -0.1 and $e^0 = 1$, we let $x_0 = 0$. To create $L(x)$ we also need to compute $f'(0)$. Since

$$f'(x) = \frac{d}{dx}e^x = e^x,$$

we have

$$f'(0) = e^0 = 1.$$

Next,

$$\begin{aligned} L(x) &= f(x_0) + f'(x_0)(x - x_0) \\ &= f(0) + f'(0)(x - 0) \\ &= 1 + 1(x - 0) \\ &= 1 + x \end{aligned}$$

and finally,

$$e^{-0.1} \approx L(0) = 1 - 0.1 = 0.9.$$

LA5. We can approximate $\ln(1.1)$ using linear approximation as follows. Let $f(x) = \ln x$ and since 1 is near 1.1 and $\ln(1) = 0$, we let $x_0 = 1$. To create $L(x)$ we also need to compute $f'(1)$. Since

$$f'(x) = \frac{d}{dx} \ln x = \frac{1}{x},$$

we have

$$f'(1) = \frac{1}{1} = 1.$$

Next,

$$\begin{aligned} L(x) &= f(x_0) + f'(x_0)(x - x_0) \\ &= f(1) + f'(1)(x - 1) \\ &= 0 + 1(x - 1) \\ &= x - 1 \end{aligned}$$

and finally,

$$\ln(1.1) \approx L(1.1) = 1.1 - 1 = 0.1.$$

LA6. We can approximate 1.9^3 using linear approximation as follows. Let $f(x) = x^3$ and since 2 is near 1.9 and $2^3 = 8$, we let $x_0 = 2$. To create $L(x)$ we also need to compute $f'(2)$. Since

$$f'(x) = \frac{d}{dx}(x^3) = 3x^2,$$

we have

$$f'(2) = 3(2^2) = 12.$$

Next,

$$\begin{aligned} L(x) &= f(x_0) + f'(x_0)(x - x_0) \\ &= f(2) + f'(2)(x - 2) \\ &= 8 + 12(x - 2) \end{aligned}$$

and finally,

$$\begin{aligned} 1.9^3 &\approx L(2) = 8 + 12(1.9 - 2) \\ &= 8 - 12(.1) = 8 - (1.2) = 6.8. \end{aligned}$$

LA7. We can approximate $\sin(1^\circ)$ using linear approximation with the following twist. In the formula

$$\frac{d}{dx} \sin(x) = \cos(x)$$

it is understood that the angle is measured in radians. Therefore, in order to use our linear approximation formula we need to restate our problem in radians as:

$$\text{Approximate } \sin\left(\frac{\pi}{180}\right).$$

We let $f(x) = \sin x$ and since 0 is near $\frac{\pi}{180}$ and $\sin(0) = 0$, we let $x_0 = 0$. To create $L(x)$ we also need to compute $f'(0)$. Since

$$f'(x) = \frac{d}{dx} \sin(x) = \cos(x),$$

we have

$$f'(0) = \cos(0) = 1.$$

Next,

$$\begin{aligned} L(x) &= f(x_0) + f'(x_0)(x - x_0) \\ &= f(0) + f'(0)(x - 0) \\ &= 0 + 1(x - 0) \\ &= x \end{aligned}$$

and finally,

$$\sin(1^\circ) = \sin\left(\frac{\pi}{180}\right) \approx L\left(\frac{\pi}{180}\right) = \frac{\pi}{180}.$$