0.1 Computing Limits Analytically

In this section, functions will be presented using formulas. We will determine the limit of a function by making an appropriate algebraic manipulations. We will factor polynomials, multiply conjugate radicals, simplify complex fractions, compute absolute values and make sign analyses.

Factor and Cancel

Factoring polynomials is an important part of finding limits analytically. A key fact that makes factoring easier is the following: if p(x) is a polynomial and p(a) = 0, then p(x) factors with (x-a) as one of the factors.

AL 1. Compute the limit: $\lim\limits_{x \to 4} \dfrac{x^2 - 16}{x^2 - 4x}$.

Plugging in the terminal value, $x = 4$, yields the indeterminate form 0/0. To find this limit analytically, we will factor the numerator and denominator and simplify the fraction. Since the polynomials in both the numerator and denominator have 4 as a **root**, they will both have $(x - 4)$ as a factor. Moreover, in the numerator we have a **difference of two perfect squares** and such a form always factors in the following way:

$$a^2 - b^2 = (a + b)(a - b).$$

Applying this general formula to our example, we get

$$x^2 - 16 = (x + 4)(x - 4).$$

Next, in the denominator, we can factor out a common factor of x from each of the terms, thereby reversing the distributive property:

$$x^2 - 4x = x(x - 4).$$

With these factorizations we can reduce the fraction and find the limit:

$$\begin{aligned}
\lim_{x\to 4}\frac{x^2-16}{x^2-4x} &= \lim_{x\to 4}\frac{(x+4)(x-4)}{x(x-4)} && \text{(factoring)}\\
&= \lim_{x\to 4}\frac{x+4}{x} && \text{(canceling)}\\
&= 8/4\\
&= 2.
\end{aligned}$$

Hence,

$$\lim_{x\to 4}\frac{x^2-16}{x^2-4x} = 2.$$

AL 2. Compute the limit: $\displaystyle\lim_{x\to 2}\frac{x^2+3x-10}{x^3-8}$.

Plugging in the terminal value $x = 2$ yields the indeterminate form $\frac{0}{0}$. Since the polynomials in both the numerator and denominator have 2 as a root, they will both have $(x-2)$ as a factor. In the numerator, the quadratic polynomial will factor $(x-2)$ times another linear factor, say $(ax+b)$. We can figure out what a and b have to be by multiplying $(x-2)(ax+b)$ and setting the result equal to the original quadratic, $x^2+3x-10$. We get $a = 1$ and $b = 5$, so

$$x^2+3x-10 = (x-2)(x+5).$$

We can use a similar strategy to factor the denominator, but the details are more complicated since the polynomial is of degree 3. Factoring out $(x-2)$ leaves a quadratic polynomial, say ax^2+bx+c. Multiplying these together and comparing with x^3-8 gives, $a = 1$, $b = 2$

and $c = 4$. We can also use the formula for the **difference of two perfect cubes**:

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2).$$

Either way, we get the factorization

$$x^3 - 8 = (x - 2)(x^2 + 2x + 4).$$

Next, with these factorizations, we have

$$\begin{aligned}
\lim_{x\to 2} \frac{x^2 + 3x - 10}{x^3 - 8} &= \lim_{x\to 2} \frac{(x-2)(x+5)}{(x-2)(x^2+2x+4)} && \text{(factoring)} \\
&= \lim_{x\to 2} \frac{x+5}{x^2+2x+4} && \text{(canceling)} \\
&= \frac{2+5}{2^2 + (2\cdot 2) + 4} \\
&= \frac{7}{12}.
\end{aligned}$$

Hence,

$$\lim_{x\to 2} \frac{x^2 + 3x - 10}{x^3 - 8} = \frac{7}{12}.$$

Conjugate Radicals

AL 3. Compute the limit:

$$\lim_{x\to 1} \frac{x - \sqrt{x}}{x - 1}.$$

The idea used in this problem is to multiply by the **conjugate radical**. Expressions of the form $a + b$ and $a - b$ are called conjugate radical expressions if either of the terms a or b (or both) contains a square root. In our problem the expression $x - \sqrt{x}$ in the numerator contains a square root symbol and hence it has a conjugate radical expression, namely $x + \sqrt{x}$. The significance of conjugate radicals is revealed when they are multiplied together. Applying the difference of two squares formula (from right to left) we see that $(x-\sqrt{x})(x+\sqrt{x}) = x^2-x$. The resulting expression no longer contains a square root symbol. Let us perform the following operation on the function in our problem:

$$\lim_{x\to 1} \frac{x-\sqrt{x}}{x-1} = \lim_{x\to 1} \left(\frac{x-\sqrt{x}}{x-1}\right) \cdot \left(\frac{x+\sqrt{x}}{x+\sqrt{x}}\right).$$

Notice that we have multiplied our original fraction by one in the form of the conjugate radical over itself. The point of undertaking this seemingly complexifying step can be seen in what follows.

$$\begin{aligned}
\lim_{x\to 1} \frac{x-\sqrt{x}}{x-1} &= \lim_{x\to 1} \frac{x-\sqrt{x}}{x-1} \cdot \frac{x+\sqrt{x}}{x+\sqrt{x}} \\
&= \lim_{x\to 1} \frac{x^2-x}{(x-1)(x+\sqrt{x})} \\
&= \lim_{x\to 1} \frac{x(x-1)}{(x-1)(x+\sqrt{x})} \\
&= \lim_{x\to 1} \frac{x}{x+\sqrt{x}} \\
&= \frac{1}{2}.
\end{aligned}$$

In this last form, we can plug in the terminal value $x = 1$ to get the final answer $\frac{1}{2}$. In conclusion, by introducing a the conjugate radical to the problem, then using the difference of two squares formula and finally factoring and canceling, we have

$$\lim_{x\to 1} \frac{x-\sqrt{x}}{x-1} = \frac{1}{2}.$$

Here is another example that uses the conjugate radical:

AL 4. Compute the limit: $\lim\limits_{x \to -3} \dfrac{x^2 - 9}{4 - \sqrt{13 - x}}$.

If we plug in $x = -3$, the numerator and denominator are both 0. The conjugate of the radical expression in the denominator is

$$4 + \sqrt{13 - x}.$$

To simplify our calculations a little bit, lets multiply the conjugates together separately:

$$(4-\sqrt{13-x})(4+\sqrt{13-x}) = 16-(13-x) = 3+x = x+3.$$

Now back to our problem:

$$\begin{aligned}
\lim_{x \to -3} \frac{x^2 - 9}{4 - \sqrt{13 - x}} &= \lim_{x \to -3} \left(\frac{x^2 - 9}{4 - \sqrt{13 - x}}\right) \cdot \left(\frac{4 + \sqrt{13 - x}}{4 + \sqrt{13 - x}}\right) \\
&= \lim_{x \to -3} \frac{(x^2 - 9)(4 + \sqrt{13 - x})}{x + 3} \\
&= \lim_{x \to -3} \frac{(x + 3)(x - 3)(4 + \sqrt{13 - x})}{x + 3} \\
&= \lim_{x \to -3} (x - 3)(4 + \sqrt{13 - x}) \\
&= (-3 - 3)(4 + \sqrt{13 - (-3)}) \\
&= (-6)(4 + 4) \\
&= -48.
\end{aligned}$$

Complex Fractions

AL 5. Compute the limit: $\lim\limits_{x \to 4} \dfrac{\frac{1}{x} - \frac{1}{4}}{x - 4}$.

Plugging in $x = 4$ yields the familiar $0/0$ indeterminate form. The algebra skills necessary to transform the function in the problem to one which allows plugging in $x = 4$ involve subtraction and division of fractions. The rules are summarized as:

$$\frac{a}{b} - \frac{c}{d} = \frac{ad - bc}{bd} \quad \text{and} \quad \frac{\frac{a}{b}}{\frac{c}{d}} = \frac{ad}{bc}.$$

Applying these to our problem, we get

$$\begin{aligned} \lim_{x \to 4} \frac{\frac{1}{x} - \frac{1}{4}}{x - 4} &= \lim_{x \to 4} \frac{\frac{4-x}{4x}}{x - 4} \\ &= \lim_{x \to 4} \frac{4 - x}{4x(x - 4)} \\ &= \lim_{x \to 4} -\frac{1}{4x} \\ &= -\frac{1}{16}. \end{aligned}$$

AL 6. Compute the limit:

$$\lim_{x \to -2} \frac{\frac{2}{x-3} + \frac{x+4}{5}}{x^2 + 5x + 6}.$$

Carefully plugging in $x = -2$ gives $\frac{0}{0}$, so we begin simplifying the complex fraction. Since this function is bulky, let's do the addition of fractions from the numerator separately:

$$\begin{aligned} \frac{2}{x - 3} + \frac{x + 4}{5} &= \frac{10 + (x - 3)(x + 4)}{5(x - 3)} \\ &= \frac{x^2 + x - 2}{5(x - 3)} \\ &= \frac{(x - 1)(x + 2)}{5(x - 3)}. \end{aligned}$$

Now back to the original problem:

$$\begin{aligned}
\lim_{x\to -2} \frac{\frac{2}{x-3}+\frac{x+4}{5}}{x^2+5x+6} &= \lim_{x\to -2} \frac{\frac{(x-1)(x+2)}{5(x-3)}}{(x+2)(x+3)} \\
&= \lim_{x\to -2} \frac{(x-1)(x+2)}{5(x-3)(x+2)(x+3)} \\
&= \lim_{x\to -2} \frac{(x-1)}{5(x-3)(x+3)} \\
&= \frac{-3}{5(-5)(1)} = \frac{3}{25}.
\end{aligned}$$

Sign Analysis

In the next few examples, we will investigate infinite limits of rational functions. These occur at points where the denominator of the rational function is approaching zero, but at the same time, the numerator is not approaching zero.

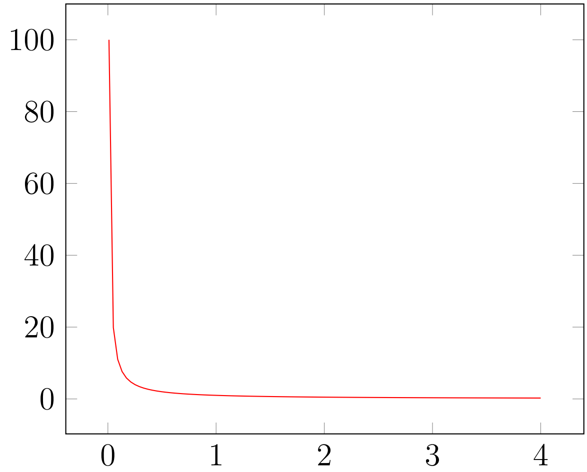
AL 7. Consider the limit: $\displaystyle\lim_{x\to 0^+} \frac{1}{x}$.

Plugging in $x = \infty$ yields the **undefined** expression $\frac{1}{0}$. As the denominator gets smaller and smaller, it will divide into the numerator more and more times, causing the fraction to "blow up". In terms of the limit, it is reasonable to expect an answer of either ∞ or $-\infty$. A sign analysis will determine which one. Since the values of x are positive as $x \to 0^+$, the values of $f(x) = \frac{1}{x}$ are also positive. Furthermore, as the values of x decrease towards zero, the values of the reciprocal, $\frac{1}{x}$ increase without bound.

Hence, we can conclude that

$$\lim_{x\to 0^+} \frac{1}{x} = \infty.$$

The geometric significance of this result is that the line $x = 0$ (the y-axis) is a vertical asymptote for the graph of the function $f(x) = \frac{1}{x}$. See the graph below.

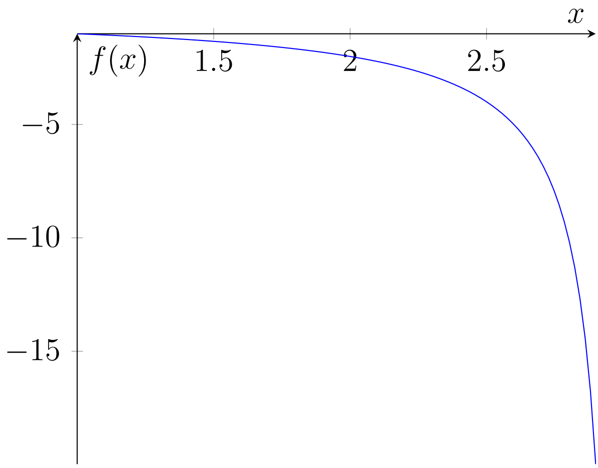


AL 8. Consider the limit: $\lim_{x\to 3^-} \dfrac{2}{x-3}$.

First we observe that plugging in the value $x = 3$ gives $\frac{2}{0}$ which is undefined and the fraction is "blowing up" in the limit. A sign analysis will tell us if the limit is $\pm\infty$. Since $x \to 3^-$, we have $x < 3$ and hence $x - 3 < 0$. We see that the numerator is 2 and the denominator is approaching 0 through negative values, and the values of $f(x) = \frac{2}{x-3}$ are negative. We conclude that

$$\lim_{x\to 3^-} \frac{2}{x-3} = -\infty.$$

This geometric significance of the result is that the line $x = 3$ is a vertical asymptote for the graph of the function $f(x) = \frac{2}{x-3}$, as shown below.



AL 9. Plugging $x = 2$ into the rational function

$$f(x) = \frac{x+1}{x-2}$$

gives the undefined expression $\frac{3}{0}$. From this information, we can conclude that the graph of the function has a vertical asymptote at $x = 2$. This means that the one-sided limits as x approaches 2 will give either ∞ or $-\infty$, i.e.,

$$\lim_{x\to 2^-} \frac{x+1}{x-2} = \infty \text{ or } -\infty.$$

and

$$\lim_{x\to 2^+} \frac{x+1}{x-2} = \infty \text{ or } -\infty.$$

To determine which, we will do a sign analysis as follows. Consider the left hand limit first:

$$\lim_{x\to 2^-} \frac{x+1}{x-2}.$$

The numerator is approaching 3, which is positive. The denominator is approaching zero which is neither positive nor negative, but since $x \to 2^-$, we know that $x < 2$

and therefore $x - 2 < 0$. Hence the denominator is negative as x approaches 2 from the left. Since a positive divided by a negative is negative, we get:

$$\lim_{x \to 2^-} \frac{x+1}{x-2} = \frac{\text{pos}}{\text{neg}} = -\infty,$$

since the choices were only ∞ and $-\infty$. Now, we will do a sign analysis on the right hand limit:

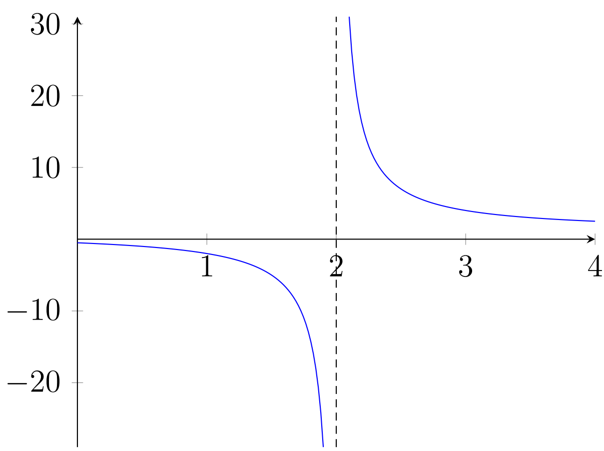
$$\lim_{x \to 2^-} \frac{x+1}{x-2}.$$

The numerator is approaching 3, which is positive. In the denominator, since $x \to 2^+$, we know that $x > 2$ and therefore $x - 2 > 0$. Hence the denominator is positive as x approaches 2 from the right. Since a positive divided by a positive is positive, we get:

$$\lim_{x \to 2^-} \frac{x+1}{x-2} = \frac{\text{pos}}{\text{pos}} = \infty,$$

since the choices were only ∞ and $-\infty$. The graph of $f(x) = \frac{x+1}{x-2}$ near $x = 2$ looks like this:

The graph of $y = \dfrac{x+1}{x-2}$



AL 10. Plugging $x = -1$ into the rational function

$$f(x) = \frac{2x}{(x+1)^3}$$

gives the undefined expression $\frac{-2}{0}$. From this information, we can conclude that the graph of the function has a vertical asymptote at $x = -1$. This means that the one-sided limits as x approaches -1 will give either ∞ or $-\infty$, i.e.,

$$\lim_{x \to -1^-} \frac{2x}{(x+1)^3} = \infty \text{ or } -\infty.$$

and

$$\lim_{x \to -1^+} \frac{2x}{(x+1)^3} = \infty \text{ or } -\infty.$$

To determine which, we will do a sign analysis as follows. Consider the left hand limit first:

$$\lim_{x \to -1^-} \frac{2x}{(x+1)^3}.$$

The numerator is approaching -2, which is negative. The denominator is approaching zero which is neither positive nor negative, but since $x \to -1^-$, we know that $x < -1$ and therefore $x+1 < 0$ and $(x+1)^3 < 0$. Hence the denominator is negative as x approaches -1 from the left. Since a negative divided by a negative is positive, we get:

$$\lim_{x \to -1^-} \frac{2x}{(x+1)^3} = \frac{\text{neg}}{\text{neg}} = \infty,$$

since the choices were only ∞ and $-\infty$. Now, we will do a sign analysis on the right hand limit:

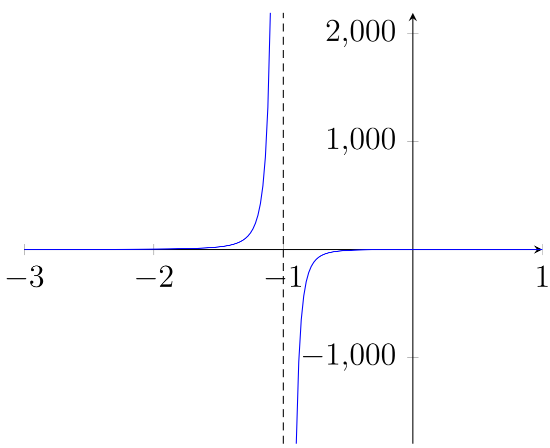
$$\lim_{x \to -1^+} \frac{2x}{(x+1)^3}.$$

The numerator is approaching -2, which is negative. In the denominator, since $x \to -1^+$, we know that $x > -1$ and therefore $x + 1 > 0$ and $(x+1)^3 > 0$. Hence the denominator is positive as x approaches -1 from the right. Since a negative divided by a positive is negative, we get:

$$\lim_{x \to -1^+} \frac{2x}{(x+1)^3} = \frac{\text{neg}}{\text{pos}} = -\infty,$$

since the choices were only ∞ and $-\infty$. Near $x = 2$, the graph looks like this:

The graph of $y = \dfrac{2x}{(x+1)^3}$



AL 11. Plugging $x = 0$ into the rational function

$$f(x) = \frac{1}{x^2}$$

gives the undefined expression $\frac{1}{0}$. From this information, we can conclude that the graph of the function has a vertical asymptote at $x = 0$. This means that the one-sided limits as x approaches 0 will give either ∞ or $-\infty$, i.e.,

$$\lim_{x\to -1^-} \frac{1}{x^2} = \infty \text{ or } -\infty.$$

and

$$\lim_{x\to -1^+} \frac{1}{x^2} = \infty \text{ or } -\infty.$$

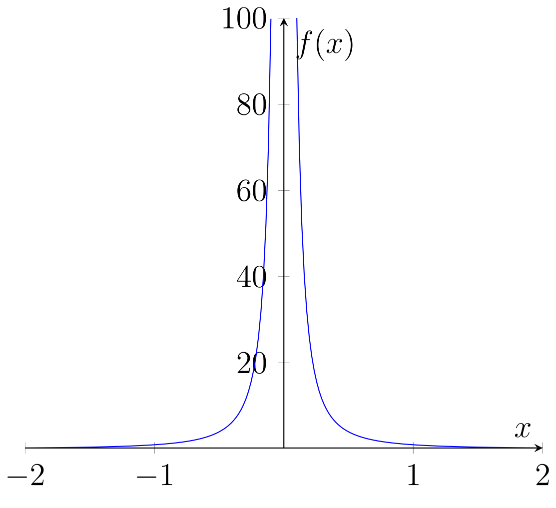
To determine which, we will do a sign analysis as follows. Because of the perfect square, we can consider both case simultaneously. The numerator is 1, which is positive and the denominator is x^2 which is positive whether $x \to 0^-$ or $x \to 0^+$. Hence

$$\lim_{x\to 0^-} \frac{1}{x^2} = \frac{\text{pos}}{\text{pos}} = \infty,$$

and

$$\lim_{x\to 0^+} \frac{1}{x^2} = \frac{\text{pos}}{\text{pos}} = \infty,$$

since the choices were only ∞ and $-\infty$. The graph of $f(x) = \frac{1}{x^2}$ near $x = 0$ looks like this:



AL 12. Compute the limit: $\lim\limits_{x\to 3^-} \dfrac{|x-3|}{x-3}$.

Plugging in $x = 3$ gives the indeterminate form $0/0$. To resolve this limit, we will do a sign analysis on the quantity in the absolute value bars, $x - 3$.

Since $x \to 3^-$, we have $x < 3$ and hence $x - 3 < 0$. Since the quantity in the absolute value bars is negative, we can compute its absolute value as follows:

$$|x-3| = -(x-3).$$

Using this in the limit, we get

$$\begin{aligned}
\lim_{x\to 3^-} \frac{|x-3|}{x-3} &= \lim_{x\to 3^-} \frac{-(x-3)}{x-3} \\
&= \lim_{x\to 3^-} (-1) \\
&= -1.
\end{aligned}$$

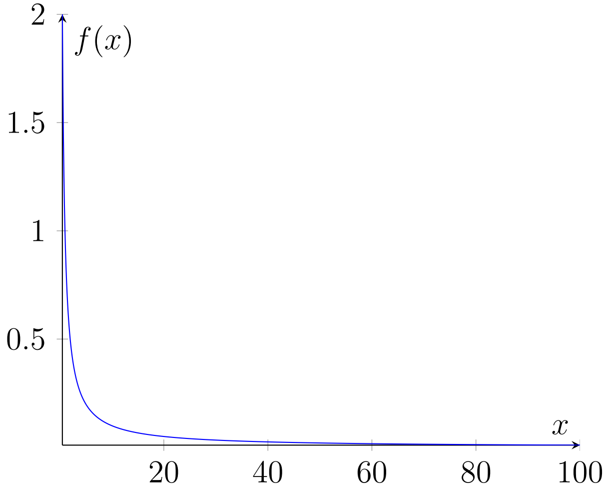
End Behavior

AL 13. Find $\lim_{x\to\infty} \dfrac{1}{x}$. We argue as follows. If x is a very large number, then $1/x$ will be a very small number, near zero. Furthermore, as x increases, $1/x$ will decrease.

Hence

$$\lim_{x\to\infty} \frac{1}{x} = 0.$$

The graph of the function $f(x) = \frac{1}{x}$ also provides evidence for this conclusion.



Lastly, the value of the limit corresponds to the horizontal asymptote of the graph, namely $y = 0$ in this example.

An important generalization, which follws from a similar analysis is

$$\lim_{x\to\pm\infty} \frac{a}{x^n} = 0$$

for any constant, a, and any positive exponent, n.

We will exploit this important fact in the next two examples.

AL 14. Compute $\lim\limits_{x\to\infty} \dfrac{3x^2+5x+2}{2x^2-x-4}$.

First note that as $x \to \pm\infty$, a polynomial, $p(x) \to \pm\infty$ according to it leading term. In this example, since the lead terms $3x^2$ and $2x^2$ both go to ∞ as $x \to \infty$, our limit has the indeterminate form $\frac{\infty}{\infty}$. To resolve this issue, we will factor out of both the numerator and the denominator the highest power of x seen in the denominator. So, in this example, we will factor x^2 from both.

In the numerator,

$$3x^2 + 5x + 2 = x^2(3 + \frac{5}{x} + \frac{2}{x^2})$$

and in the denominator,

$$2x^2 - x - 4 = x^2(2 - \frac{1}{x} - \frac{4}{x^2}).$$

With these factorizations, our limit becomes

$$\begin{aligned}
\lim_{x\to\infty} \frac{3x^2 + 5x + 2}{2x^2 - x - 4} &= \lim_{x\to\infty} \frac{x^2(3 + \frac{5}{x} + \frac{2}{x^2})}{x^2(2 - \frac{1}{x} - \frac{4}{x^2})} \\
&= \lim_{x\to\infty} \frac{3 + \frac{5}{x} + \frac{2}{x^2}}{2 - \frac{1}{x} - \frac{4}{x^2}} \\
&= \frac{3 + 0 + 0}{2 - 0 - 0} \\
&= \frac{3}{2}.
\end{aligned}$$

The result of this limit means that the line $y = 3/2$ is a horizontal asymptote for the graph of $y = \dfrac{3x^2 + 5x + 2}{2x^2 - x - 4}$ on the right end.

AL 15. Compute $\displaystyle\lim_{x\to-\infty} \frac{4x^2 - 3x - 6}{x^3 + 8x^2 - 3x + 7}$.

First note that as $x \to \pm\infty$, a polynomial, $p(x) \to \pm\infty$ according to it leading term. In this example, the lead terms $4x^2$ and x^3 go to ∞ and $-\infty$ respectively, as

$x \to \infty$. Hence, our limit has the indeterminate form $\frac{\infty}{-\infty}$. To resolve this issue, we will factor out of both the numerator and the denominator the highest power of x seen in the denominator. So, in this example, we will factor x^3 from both. In the numerator,

$$4x^2 - 3x - 6 = x^3(\frac{4}{x} - \frac{3}{x^2} - \frac{6}{x^3})$$

and in the denominator,

$$x^3 + 8x^2 - 3x + 7 = x^3(1 + \frac{8}{x} - \frac{3}{x^2} + \frac{7}{x^3}).$$

With these factorizations, our limit becomes

$$\begin{aligned}
\lim_{x\to-\infty} \frac{4x^2 - 3x - 6}{x^3 + 8x^2 - 3x + 7} &= \lim_{x\to-\infty} \frac{x^3(\frac{4}{x} - \frac{3}{x^2} - \frac{6}{x^3})}{x^3(1 + \frac{8}{x} - \frac{3}{x^2} + \frac{7}{x^3})} \\
&= \lim_{x\to-\infty} \frac{\frac{4}{x} - \frac{3}{x^2} - \frac{6}{x^3}}{1 + \frac{8}{x} - \frac{3}{x^2} + \frac{7}{x^3}} \\
&= \frac{0 - 0 - 0}{1 + 0 - 0 + 0} \\
&= \frac{0}{1} \\
&= 0.
\end{aligned}$$

The result of this limit means that the line $y = 0$ (the x-axis) is a horizontal asymptote for the graph of $y = \dfrac{4x^2 - 3x - 6}{x^3 + 8x^2 - 3x + 7}$ on the right end.