## 0.1 L'Hopital's Rule

L'Hopital's Rule uses the derivative to help us find limits involving indeterminate forms. The main indeterminate forms we will discuss are $\frac{0}{0}, \frac{\infty}{\infty}, 0 \cdot \infty, 1^{\infty}$ and $0^0$. We begin with the fractional forms.

**L'Hopital's Rule**

If $\lim\limits_{x \to c} \dfrac{f(x)}{g(x)} = \dfrac{0}{0}$ or $\dfrac{\infty}{\infty}$ then $\lim\limits_{x \to c} \dfrac{f(x)}{g(x)} = \lim\limits_{x \to c} \dfrac{f'(x)}{g'(x)}$ provided the latter exists.

In the above statement, $x \to c$ can be replaced by a one-sided limit and $c$ can be $\pm\infty$. Also the fraction $\frac{\infty}{\infty}$ is shorthand for $\frac{\pm\infty}{\pm\infty}$.

### The $\dfrac{0}{0}$ case

LR 1. Compute the limit: $\lim\limits_{x \to 0} \dfrac{\sin(x)}{x}$.

Plugging in the terminal value, $x = 0$, yields the indeterminate form $0/0$, so L'Hopital's rule applies.

We have

$$\lim_{x \to 0} \frac{\sin(x)}{x} = \lim_{x \to 0} \frac{\cos(x)}{1} = 1.$$

LR 2. Compute the limit: $\lim\limits_{x \to 0} \dfrac{e^x - 1}{x}$.

Plugging in the terminal value, $x = 0$, yields the indeterminate form $0/0$, so L'Hopital's rule applies.

We have

$$\lim_{x \to 0} \frac{e^x - 1}{x} = \lim_{x \to 0} \frac{e^x}{1} = 1.$$

LR 3. Compute the limit: $\lim\limits_{x \to 2} \dfrac{x^3 - 8}{x^5 - 32}$.

Plugging in the terminal value, $x = 2$, yields the indeterminate form $0/0$, so L'Hopital's rule applies.

We have

$$\lim_{x\to 2} \frac{x^3-8}{x^5-32} = \lim_{x\to 2} \frac{3x^2}{5x^4} = \frac{12}{80} = \frac{3}{20}.$$

Sometimes we have to use L'Hopital's Rule more than once.

LR 4. Compute the limit: $\lim\limits_{x\to 0} \dfrac{e^x - x - 1}{x^2}$.

Plugging in the terminal value, $x = 0$, yields the indeterminate form $0/0$, so L'Hopital's rule applies. We have

$$\lim_{x\to 0} \frac{e^x - x - 1}{x^2} = \lim_{x\to 0} \frac{e^x - 1}{2x} = \frac{0}{0}.$$

Applying L'Hopital's Rule again gives

$$\lim_{x\to 0} \frac{e^x - 1}{2x} = \frac{e^x}{2} = \frac{1}{2}.$$

Hence $\lim\limits_{x\to 0} \dfrac{e^x - x - 1}{x^2} = \dfrac{1}{2}$.

LR 5. Compute the limit: $\lim\limits_{x\to 0} \dfrac{\tan(x) - x}{x^3}$.

Plugging in the terminal value, $x = 0$, yields the indeterminate form $0/0$, so L'Hopital's rule applies.

We have

$$\lim_{x\to 0} \frac{\tan(x) - x}{x^3} = \lim_{x\to 0} \frac{\sec^2(x) - 1}{3x^2} = \frac{0}{0}.$$

Applying L'Hopital's Rule again gives

$$\lim_{x\to 0} \frac{\sec^2(x) - 1}{3x^2} = \lim_{x\to 0} \frac{2\sec^2(x)\tan(x)}{6x} = \frac{0}{0}.$$

We need to apply L'Hopital's Rule again, but first, the numerator is complicated and

$$\lim_{x\to 0} \sec(x) = 1 \neq 0$$

so we take a simplifying step before applying the rule.

$$\lim_{x\to 0}\frac{2\sec^2(x)\tan(x)}{6x} = \lim_{x\to 0}\sec^2(x)\cdot\lim_{x\to 0}\frac{2\tan(x)}{6x}$$
$$= 1\cdot\lim_{x\to 0}\frac{2\sec^2(x)}{6} = \frac{2}{6} = \frac{1}{3}.$$

Hence $\lim_{x\to 0}\dfrac{\tan(x)-x}{x^3} = \dfrac{1}{3}.$

### The $\dfrac{\infty}{\infty}$ case

These are handled the same way as the 0/0 case above.

**LR 6.** Compute the limit: $\lim_{x\to\infty}\dfrac{x^2}{e^x}.$

As $x$ approaches $\infty$ we get the indeterminate form $\frac{\infty}{\infty}$ so L'Hopital's Rule applies.

We have

$$\lim_{x\to\infty}\frac{x^2}{e^x} = \lim_{x\to\infty}\frac{2x}{e^x} = \frac{\infty}{\infty}.$$

Applying L'hopital again, we get

$$\lim_{x\to\infty}\frac{2x}{e^x} = \lim_{\to\infty}\frac{2}{e^x} = 0.$$

Hence $\lim_{x\to\infty}\frac{x^2}{e^x} = 0$. This limit can be generalized as follows:

$$\lim_{x\to\infty}\frac{x^n}{e^x} = 0$$

for any exponent $n = 1, 2, 3, \ldots$.

This general result comes from using L'Hopital's Rule $n$ times, yielding

$$\lim_{x\to\infty}\frac{n!}{e^x} = 0$$

where $n! = n(n-1)(n-2)\cdot\cdots\cdot 3\cdot 2\cdot 1$.

The interpretation of this limit is that the exponential function $e^x$ grows faster than any power of $x$ as $x\to\infty$.

**LR 7.** Compute the limit: $\lim_{x\to\infty}\dfrac{\sqrt{x}}{\ln(x)}.$

As $x\to\infty$ we get $\infty/\infty$, so L'Hopital's Rule applies.

We have:

$$\lim_{x\to\infty} \frac{\sqrt{x}}{\ln(x)} = \lim_{x\to\infty} \frac{\left(\frac{1}{2\sqrt{x}}\right)}{\left(\frac{1}{x}\right)}$$

which simplifies to

$$\lim_{x\to\infty} \frac{x}{2\sqrt{x}} = \lim_{x\to\infty} \frac{\sqrt{x}}{2} = \infty.$$

Hence, $\lim_{x\to\infty} \frac{\sqrt{x}}{\ln(x)} = \infty$.

The interpretation of this limit is that $\sqrt{x}$ goes to $\infty$ faster than $\ln(x)$ as $x \to \infty$.

Other indeterminate forms.

L'Hopital's Rule requires a fractional indeterminate form such as $0/0$ or $\infty/\infty$, but we can use it to handle other indeterminate forms by rewriting the problem in the format of a fraction.

## The $0 \cdot \infty$ case

**LR 8.** Compute the limit: $\displaystyle\lim_{x\to 0^+} x^2 \ln(x)$.

As $x \to 0^+$ we get $0 \cdot (-\infty)$ which is an indeterminate form, but L'Hopital's Rule does not apply in this situation. We must rewrite the problem as a fraction, in the following way:

$$\lim_{x\to 0^+} \frac{\ln(x)}{x^{-2}}.$$

Notice that this is equivalent to the original problem since

$$x^2 = \frac{1}{x^{-2}}.$$

Also note that $x^{-2} = \frac{1}{x^2} \to \infty$ as $x \to 0^+$.

Now, we can use L'Hopital's Rule because

$$\lim_{x\to 0+} \frac{\ln(x)}{x^{-2}} = \frac{-\infty}{\infty}.$$

We get

$$\lim_{x\to 0^+} \frac{\ln(x)}{x^{-2}} = \lim_{x\to\infty} \frac{1/x}{-2x^{-3}}$$

which simplifies to

$$\lim_{x\to 0^+} -\frac{x^2}{2} = 0.$$

Hence,

$$\lim_{x\to\infty} \frac{\ln(x)}{x^{-2}} = 0.$$