0.1 The Indefinite Integral

The notation for the **indefinite integral** is $\int f(x)\ dx$ and it is read "the integral of $f(x)$ dx." To compute an indefinite integral we need to find all functions $F(x)$ whose derivative is $f(x)$. The process of computing an integral is called integration, or anti-differentiation. Since any two functions that have the same derivative differ by a constant, if we can find one example of a function $F(x)$ whose derivative is $f(x)$, i.e., $F'(x) = f(x)$, then we can write

$$\int f(x)\ dx = F(x) + C$$

where C is any constant. The symbol $\int$ is called the integral sign, the function $f(x)$ is called the integrand and C is called the constant of integration.

Examples of Basic Indefinite Integrals

A basic indefinite integral is one that can be computed either by recognizing the integrand as the derivative of a familiar function or by reversing the power rule for derivatives.

II 1. $\int \cos(x)\ dx = \sin(x) + C$, because the derivative of $\sin(x)$ is $\cos(x)$.

II 2. $\int \sec^2(x)\ dx = \tan(x) + C$, because the derivative of $\tan(x)$ is $\sec^2(x)$.

II 3. $\int \sec(x)\tan(x)\ dx = \sec(x) + C$, because the derivative of $\sec(x)$ is $\sec(x)\tan(x)$.

II 4. $\int e^x\ dx = e^x + C$, because the derivative of e^x is e^x.

II 5. $\int 2x\ dx = x^2 + C$, because the derivative of x^2 is $2x$.

II 6. $\int \frac{1}{2\sqrt{x}}\ dx = \sqrt{x} + C$, because the derivative of $\sqrt{x}$ is $\frac{1}{2\sqrt{x}}$.

II 7. $\int \frac{1}{1+x^2}\ dx = \tan^{-1}(x) + C$, because the derivative of $\tan^{-1}(x)$ is $\frac{1}{1+x^2}$.

II 8. $\int \frac{1}{\sqrt{1-x^2}}\ dx = \sin^{-1}(x) + C$, because the derivative of $\sin^{-1}(x)$ is $\frac{1}{\sqrt{1-x^2}}$.

II 9. $\int -\sin(x)\ dx = \cos(x) + C$, because the derivative of $\cos(x)$ is $-\sin(x)$.

II 10. $\int 2^x \ln(2)\ dx = 2^x + C$, because the derivative of 2^x is $2^x \ln(2)$.

Power Rule for Indefinite Integrals

$$\int x^n\ dx = \frac{x^{n+1}}{n+1} + C$$

where n is any number except -1.

Examples of the Power Rule for Indefinite Integrals

PR 1. To compute $\int x^3\ dx$ we use the power rule with $n = 3$ and we get

$$\int x^3\ dx = \frac{x^4}{4} + C.$$

The answer can be confirmed by observing that the derivative of $\frac{x^4}{4} + C$ is x^3 (using the constant multiple rule and the power rule for derivatives).

PR 2. To compute $\int 1\ dx$ we use the power rule with $n = 0$ ($x^0 = 1$) and we get

$$\int 1\ dx = \int x^0\ dx = \frac{x^1}{1} + C = x + C.$$

The answer can be confirmed by observing that the derivative of $x + C$ is 1

PR 3. To compute $\int x\ dx$ we use the power rule with $n = 1$ and we get

$$\int x\ dx = \frac{x^2}{2} + C.$$

The answer can be confirmed by observing that the derivative of $\frac{x^2}{2} + C$ is x (using the constant multiple rule and the power rule for derivatives).

PR 4. To compute $\int \sqrt{x}\ dx$ we rewrite $\sqrt{x}$ as $x^{1/2}$ and we use the power rule with $n = 1/2$ and we get

$$\begin{aligned}\int \sqrt{x}\ dx &= \int x^{1/2}\ dx \\ &= \frac{x^{3/2}}{3/2} + C \\ &= \tfrac{2}{3}x^{3/2} + C \\ &= \tfrac{2}{3}\sqrt{x^3} + C \\ &= \tfrac{2}{3}x\sqrt{x} + C.\end{aligned}$$

PR 5. To compute $\int \frac{1}{x^3}\ dx$ we rewrite $\frac{1}{x^3}$ as x^{-3} and use the power rule with $n = -3$. We get

$$\begin{aligned}\int \frac{1}{x^3}\ dx &= \int x^{-3}\ dx \\ &= \frac{x^{-2}}{-2} + C \\ &= -\tfrac{1}{2}x^{-2} + C \\ &= -\tfrac{1}{2} \cdot \frac{1}{x^2} + C \\ &= -\frac{1}{2x^2} + C.\end{aligned}$$

PR 6. To compute $\int \frac{1}{\sqrt{x}}\,dx$ we rewrite $\frac{1}{\sqrt{x}}$ as $x^{-1/2}$ and use the power rule with $n=-1/2$. We get

$$\begin{aligned}\int \frac{1}{\sqrt{x}}\,dx &= \int x^{-1/2}\,dx\\ &= \frac{x^{1/2}}{1/2}+C\\ &= 2x^{1/2}+C\\ &= 2\sqrt{x}+C.\end{aligned}$$

Special Case

The power rule does not work when $n=-1$ so we consider this as a special case:

$$\int x^{-1}\,dx = \int \frac{1}{x}\,dx = \ln|x|+C.$$

We can verify this by noting that the derivative of $\ln|x|+C$ is $\dfrac{1}{x}$.

We can handle sums, differences and constant multiples in indefinite integrals the exact same way we handle them in differentiation.

Constant Multiples

CM 1. $\displaystyle\int 5\cos(x)\,dx = 5\sin(x)+C.$

CM 2. $\displaystyle\int 3\sec^2(x)\,dx = 3\tan(x)+C.$

CM 3. $\displaystyle\int 2e^x\,dx = 2e^x+C.$

CM 4. $\displaystyle\int \frac{4}{1+x^2}\,dx = 4\tan^{-1}(x)+C.$

CM 5. $\displaystyle\int \frac{7}{12x}\,dx = \frac{7}{12}\ln|x|+C.$

CM 6. $\displaystyle\int 4x^7\,dx = 4\cdot\frac{x^8}{8}+C = \frac{x^8}{2}.$

CM 7. $$\begin{aligned}\int \frac{3}{5x^2}\,dx &= \int \frac{3}{5}x^{-2}\,dx\\ &= \frac{3}{5}\cdot\frac{x^{-1}}{-1}+C\\ &= -\frac{3}{5}\cdot\frac{1}{x}+C\\ &= -\frac{3}{5x}+C.\end{aligned}$$

CM 8. $\displaystyle\int 4\sin(x)\ dx = -4\cos(x) + C.$

CM 9. $\displaystyle\int 3\csc(x)\cot(x)\ dx = -3\csc(x) + C.$

CM 10.
$$\begin{aligned}\int 4\sqrt[3]{x}\ dx &= \int 4x^{1/3}\ dx \\ &= 4\cdot\frac{x^{4/3}}{4/3} + C \\ &= 4\cdot\frac{3}{4}\cdot x^{4/3} + C \\ &= 3x^{4/3} + C \\ &= 3\sqrt[3]{x^4} + C \\ &= 3x\sqrt[3]{x} + C.\end{aligned}$$

Sums

SR 1. $\int \cos(x) + e^x\ dx = \sin(x) + e^x + C.$

SR 2. $\int x + 2\sec^2(x)\ dx = \frac{x^2}{2} + 2\tan(x) + C.$

SR 3. $\int x^2 + \frac{1}{x^2}\ dx = \int x^2 + x^{-2}\ dx = \frac{x^3}{3} + \frac{x^{-1}}{-1} + C = \frac{x^3}{3} - \frac{1}{x} + C$

SR 4. $\int \frac{x+2}{x}\ dx = \int \frac{x}{x} + \frac{2}{x}\ dx = \int 1 + \frac{2}{x}\ dx = x + 2\ln|x| + C.$

SR 5.
$$\begin{aligned}\int \sqrt{x} + \sqrt[3]{x}\ dx &= \int x^{1/2} + x^{1/3}\ dx \\ &= \frac{x^{3/2}}{3/2} + \frac{x^{4/3}}{4/3} + C \\ &= \tfrac{2}{3}x^{3/2} + \tfrac{3}{4}x^{4/3} + C \\ &= \tfrac{2}{3}\sqrt{x^3} + \tfrac{3}{4}\sqrt[3]{x^4} + C \\ &= \tfrac{2}{3}x\sqrt{x} + \tfrac{3}{4}x\sqrt[3]{x} + C.\end{aligned}$$

SR 6. $\int x^2 + 3x + 4\ dx = \frac{x^3}{3} + \frac{3x^2}{2} + 4x + C.$

Differences

DR 1. $\int (1 - e^x)\ dx = x - e^x + C.$

DR 2. $\int (x^2 - x)\ dx = \frac{x^3}{3} - \frac{x^2}{2} + C.$

DR 3. $\int [3\cos(x) - 2\sin(x)]\ dx = 3\sin(x) + 2\cos(x) + C.$

DR 4. $\int (3x^2 - \frac{3}{x} - \frac{1}{1+x^2})\ dx = x^3 - 3\ln|x| - \tan^{-1}(x) + C.$

DR 5. $\int [4\csc(x)(\csc(x) - \cot(x))]\ dx = \int [4\csc^2(x) - 4\csc(x)\cot(x)]\ dx = -4\cot(x) + 4\csc(x) + C = 4\csc(x) - 4\cot(x) + C.$