0.1 Increasing and Decreasing Functions

In this section, we use the derivative to determine intervals on which a given function is increasing or decreasing. We can then also determine the location of local extremes of the function.

Definitions of Increasing and Decreasing

A function $f(x)$ is called **increasing** on an interval I if given any two numbers, x_1 and x_2 in I such that $x_1 < x_2$, we have $f(x_1) < f(x_2)$.
Similarly, a function $f(x)$ is called **decreasing** on an interval I if given any two numbers, x_1 and x_2 in I such that $x_1 < x_2$, we have $f(x_1) > f(x_2)$.

As an example of increasing, the function $f(x) = x^2$ is increasing on the interval $(0, \infty)$ since if $0 < x_1 < x_2$ then $(x_1)^2 < (x_2)^2$.

As an example of decreasing, the function x^2 is decreasing on the interval $(-\infty, 0)$ since if $x_1 < x_2 < 0$ then $(x_1)^2 > (x_2)^2$.

To help see this, make a rough sketch of the function $y = x^2$.

To help us determine where more complicated functions are either increasing or decreasing, we have the following theorem.

Increasing/Decreasing Theorem

If $f'(x) > 0$ on an interval I, then $f(x)$ is increasing on the interval I. If $f'(x) < 0$ on an interval I, then $f(x)$ is decreasing on the interval I.

Proof (Increasing case): Let x_1 and x_2 be in the interval I with $x_1 < x_2$. Then since f is differentiable on the closed interval $[x_1, x_2]$, it is also continuous there. Hence we can apply the MVT on to $f(x)$ on $[x_1, x_2]$ and conclude that

$$f(x_2) - f(x_1) = f'(c)(x_2 - x_1).$$

Both factors on the right hand side are positive, hence $f(x_2) - f(x_1)$ is positive and $f(x_2) > f(x_1)$. Hence $x_1 < x_2$implies $f(x_1) < f(x_2)$ which means that $f(x)$ is increasing on I.

We now use this theorem to determine intervals on which a given function is either increasing or decreasing.

ID 1 Determine intervals on which $f(x) = \dfrac{x^3}{3} - \dfrac{x^2}{2} - 6x + 2$ is increasing and decreasing. According to the theorem, we must determine where $f'(x)$ is either positive or negative. To do this, it is often easiest to first determine where $f'(x) = 0$ or $f'(x)$ is undefined. In this example,

$$f'(x) = x^2 - x - 6$$

which exists for all x. We solve the equation

$$f'(x) = x^2 - x - 6 = 0$$

which yields

$$(x + 2)(x - 3) = 0$$

and hence, $x = -2$ or $x = 3$. These two x-values break the real number line into three open intervals: $(-\infty, -2), (-2, 3)$ and $(3, \infty)$. On each of these intervals, f' will be either strictly positive or strictly negative. To determine which, we will use test points. For the interval, $(-\infty, -2)$ consider $f'(-3) = (-3)^2 - (-3) - 6 = 6 > 0$. This means that $f'(x) > 0$ for every x value in the interval $(-\infty, -2)$ and by the theorem, $f(x)$ is increasing on this interval.

Next, for the interval $(-2,3)$, consider $f'(0) = (0)^2 - (0) - 6 = -6 < 0$. This means that $f'(x) < 0$ for every x value in the interval $(-2,3)$ and by the theorem, $f(x)$ is decreasing on this interval.

Finally, for the interval $(3,\infty)$, consider $f'(4) = (4)^2 - (4) - 6 = 10 > 0$. This means that $f'(x) > 0$ for every x value in the interval $(3,\infty)$ and by the theorem, $f(x)$ is increasing on this interval.

The work that was done in the previous example can actually give us slightly more information about $f(x)$. We can determine the **local extremes** of $f(x)$.

Definition: We say that $f(x)$ has a **local maximum** at $x = x_0$ if there is an open interval I containing x_0 such that $f(x) \leq f(x_0)$ for all values of x in I. Similarly, we say that $f(x)$ has a **local minimum** at $x = x_0$ if there is an open interval I containing x_0 such that $f(x) \geq f(x_0)$ for all values of x in I.

First Derivative Test for Local Extremes: If $f'(x)$ changes sign at a critical number x_0, then $f(x)$ has a local extreme at $x = x_0$. More specifically, if the sign changes from positive to negative then the local extreme is a max and if the sign changes from negative to positive, then the local extreme is a min.

ID 2 In the previous example, we determined that -2 and 3 were critical numbers for the function $f(x) = \dfrac{x^3}{3} - \dfrac{x^2}{2} - 6x + 2$ and that $f(x)$ was increasing on the interval $(-\infty,-2)$, decreasing on $(-2,3)$ and increasing on $(3,\infty)$. By the First Derivative Test, we can conclude that $f(x)$ has a local maximum at $x = -2$ and a local minimum at $x = 3$.

ID 3 $f(x) = xe^{-2x}$.