0.1 Intermediate Value Theorem

In this section we discuss an important theorem related to continuous functions. Before we present the theorem, lets consider two real life situations and observe an important difference in their behavior. First, consider the ambient temperature as a and second, consider the amount of money in a bank account, both as a functions of time.

First, suppose that the temperature is 65° at 8am and then suppose it is 75° at noon. Because of the continuous nature of temperature variation, we can be sure that at some time between 8am and noon the temperature was exactly 70°. Can we make a similar claim about money in a bank account? Suppose the account has \$65 in it at 8am and then it has \$75 in it at noon. Did it have exactly \$70 in it at some time between 8 am and noon? We cannot answer that question with any certainty from the given information. On one hand, it is possible that a \$10 deposit was made at 11am and so the total in the bank would have jumped from \$65 dollars to \$75 without ever being exactly \$70. On the other hand, it is possible that the \$10 was added in \$5 increments, like \$5 at 10 am and another \$5 at 11 am. In this case, the account did have exactly \$70 in it at some time, namely 10am. Again, in the first case, we can say with certainty that the temperature was exactly 70 degrees at some time between 8am and noon but in the second case we cannot be sure whether there was ever exactly \$70 dollars in the account between 8am and noon. The fundamental reason why we can make certain conclusions in the first case but cannot in the second, is that temperature varies continuously, whereas money in a bank account does not. When a quantity is known to vary continuously, then if the quantity is observed to have different values at different times then we can conclude that the quantity took on any value between these two at some time between our two observations. Mathematically, this property is stated in the Intermediate Value Theorem: Suppose a function $f(x)$ is continuous on an interval $[a, b]$ and suppose I is a value between $f(a)$ and $f(b)$. Then there is some input, c between a and b such that $f(c) = I$. The value I is called an intermediate value. The IVT therefore can be re-stated by saying that a continuous function takes on all of its intermediate values on an interval.

If a function is not continuous on an interval, then we cannot be sure whether or not it takes on a particular intermediate value.

Examples of the IVT

IV 1. To show that the function $f(x) = x^3$ takes on the value 10 somewhere between $x = 2$ and $x = 3$, we observe that since the function is a polynomial, it is continuous on the interval $(-\infty, \infty)$ and hence it is continuous on the sub-interval, $[2, 3]$. Next, observe that $f(2) = 2^3 = 8$ and $f(3) = 3^3 = 27$ so that 10 is an intermediate value:

$$8 < 10 < 27.$$

All of the hypotheses of the Intermediate Value Theorem are satisfied so we can now conclude that there exists a value c between 2 and 3 such that $f(c) = c^3 = 10$.

IV 2. To show that the function $f(x) = e^x$ takes on the value 2 somewhere between $x = 0$ and $x = 1$, we observe that the function $f(x) = e^x$ is continuous on the interval $(-\infty, \infty)$ and hence it is continuous on the sub-interval, $[0, 1]$. Next, observe that $f(0) = e^0 = 1$ and $f(1) = e^1 = e \approx 2.7$ so that 2 is an intermediate value:

$$1 < 2 < e.$$

All of the hypotheses of the Intermediate Value Theorem are satisfied so we can now conclude that there exists a value c between 0 and 1 such that $f(c) = e^c = 2$.

IV 3. To show that the function $f(x) = x^4 - 3x - 7$ has a root in the interval $(-2, 1)$ we use the IVT. Note that having a root means that $f(x)$ takes on the value 0. Since $f(x)$ is a polynomial, it is continuous on the interval $[-2, 1]$. Plugging in the endpoints shows that 0 is an intermediate value:

$$f(-2) = (-2)^4 - 3(-2) - 7 = 16 + 6 - 7 = 15 > 0$$

and

$$f(1) = (1)^4 - 3(1) - 7 = 1 - 3 - 7 = -9 < 0.$$

By the IVT, we can conclude that there exists a value c between $x = -2$ and $x = 1$ such that $f(c) = 0$, i.e., $f(x)$ has a root in the interval $(-2, 1)$.

IV 4. We can use the IVT to prove that the equation $\cos(x) = x$ has a solution between $x = 0$ and $x = \frac{\pi}{2}$. Let

$$f(x) = x - \cos(x).$$

Then $f(x)$ is the difference between two functions which are each continuous on the interval $(-\infty, \infty)$. Hence $f(x)$ is also continuous on the interval $(-\infty, \infty)$. For our purposes, it is sufficient to note that $f(x)$ is continuous on the interval $[0, \frac{\pi}{2}]$. Next we compute $f(0)$ and $f(\frac{\pi}{2})$ and show that 0 is an intermediate value:

$$f(0) = 0 - \cos(0) = 0 - 1 = -1 < 0$$

and

$$f\left(\frac{\pi}{2}\right) = \frac{\pi}{2} - \cos\left(\frac{\pi}{2}\right) = \frac{\pi}{2} - 0 = \frac{\pi}{2} > 0.$$

By the IVT, there exists a value c in the interval $(0, \frac{\pi}{2})$ such that $f(c) = 0$. Finally, since $f(c) = 0$, we have $c - \cos(c) = 0$ which can be rewritten $\cos(c) = c$.

IV 5. Repeated use of the IVT can help us to approximate solutions to equations. In this example, we will use the Bisection Method to approximate a root of a polynomial. Let

$$p(x) = x^3 - 3x^2 + 4x - 6.$$

We would like to approximate a root of $p(x)$. We begin by noting that

$$p(0) = -6 < 0 \text{ and } p(8) = 346 > 0.$$

Since polynomials are continuous for all values of x, we can apply the IVT to a polynomial function on any interval. In this case, the IVT tells us that $p(x)$ has a root somewhere in the interval $(0, 8)$. To implement the Bisection Method, we now determine the midpoint of this interval: $(0 + 8)/2 = 4$ and we plug this in to $p(x)$:

$$p(4) = 14 > 0.$$

Combining this with

$$p(0) < 0,$$

we can use the IVT to conclude that $p(x)$ has a root on the interval $(0, 4)$. This interval is half of the original interval- the original interval has been bisected. We can do this again. The midpoint of the interval $(0, 4)$ is $x = 2$. We plug this into $p(x)$:

$$p(2) = -2 > 0.$$

Combining this with

$$p(4) > 0,$$

we can use the IVT to conclude that $p(x)$ has a root on the interval $(2, 4)$. This interval is half of the previous interval- the previous interval has been bisected. We do this once again, noting that $x = 3$ is the midpoint of the interval $(2, 4)$:

$$p(3) = 6 > 0.$$

Combining this with

$$p(2) < 0,$$

we can use the IVT to conclude that $p(x)$ has a root on the interval $(2, 3)$. Our approximation of the root is $x = 2.5$ which is the midpoint of this interval. Our error is then no more than 0.5 units, which is half the width of the interval. We will stop here, but the method could theoretically be continued indefinitely giving a better approximation to the root each time.