0.1 The Fundamental Theorem of Calculus, Part II

The second part of the FTC tells us the derivative of an area function $A(x)$. Its remarkable conclusion states that the integrand itself is the derivative in question. This conclusion establishes the theory of the existence of antiderivatives. Because of it, we know that every continuous function $f(x)$ has an anti-derivative, namely its corresponding area function:

$$A(x) = \int_a^x f(t)\ dt.$$

FTC II

If $f(x)$ is continuous on the closed interval $[a, b]$ then

$$A'(x) = \frac{d}{dx}\int_a^x f(t)\ dt = f(x),$$

for any value of x in the interval $[a, b]$.

FTCII 1. Let $f(x) = \sin(x^2)$. This function is continuous for all x. By the FTC, part II we can say that for any number a and any value of x,

$$\frac{d}{dx}\int_a^x \sin(t^2)\ dt = \sin(x^2).$$

This example asserts that the continuous function $f(x) = \sin(x^2)$ has an antiderivative. Finding another form- a familiar form- is an impossible task, despite the simple nature of the function itself. Thus the theory of antidifferentiation is much richer than the theory of differentiation, since such a simple function can be differentiated easily (using the chain rule).

FTCII 2. $\displaystyle\frac{d}{dx}\int_0^x \cos(t)\ dt = \cos(x)$, by the FTC, part II. Note that this equation can be verified easily by computing the integral and then taking the derivative of the result:

$$\frac{d}{dx}\int_0^x \cos(t)\ dt = \sin(x) - \sin(0) = \sin(x)$$

and the derivative of $\sin(x)$ is $\cos(x)$. Note also in this example that the function $\int_0^x \cos(t)\ dt$ has another, familiar form, namely $\sin(x)$.

FTCII 3. $\displaystyle\frac{d}{dx}\int_1^x e^{2t}\sqrt{1+t^3}\ dt = e^{2x}\sqrt{1+x^3}$, by the FTC, part II.

FTCII 4. $\displaystyle\frac{d}{dx}\int_x^0 f(t)\ dt = -f(x)$ since

$$\int_x^0 f(t)\ dt = -\int_0^x f(t)\ dt.$$

FTCII 5. $\displaystyle\frac{d}{dx}\int_0^{x^2} e^{2t}\sqrt{1+t^3}\ dt = 2xe^{2x^2}\sqrt{1+x^6}$ by the chain rule. To see this, let

$$A(x) = \frac{d}{dx}\int_0^{x} e^{2t}\sqrt{1+t^3}\ dt$$

then the function in the problem is $A(x^2)$. By the chain rule, the derivative of $A(x^2)$ is $A'(x^2)\cdot 2x$. Finally, since $A'(x) = e^{2x}\sqrt{1+x^3}$, we have

$$A'(x^2) = e^{2(x^2)}\sqrt{1+(x^2)^3} = e^{2x^2}\sqrt{1+x^6}$$

and

$$\frac{d}{dx}\int_0^{x^2} e^{2t}\sqrt{1+t^3}\ dt = e^{2x^2}\sqrt{1+x^6}\cdot 2x.$$

FTCII 6. $\displaystyle\frac{d}{dx}\int_{2x}^{0}\tan(3t)\ln(t^2)\ dt = -2\tan(6x)\ln(4x^2)$. Note that the negative sign comes from switching the limits of integration and the 2 comes from the chain rule ($\frac{d}{dx}2x = 2$).