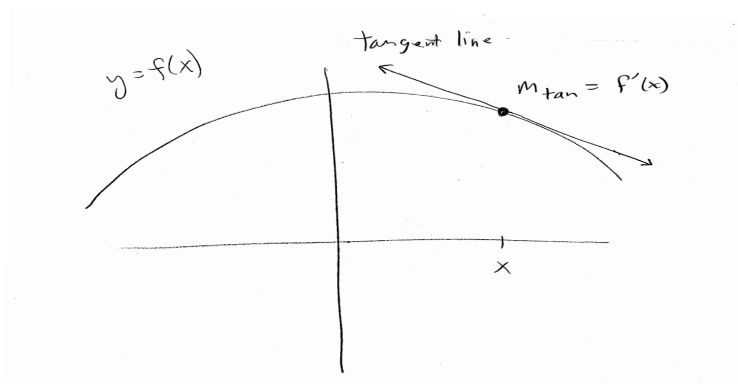
# 0.1 The Derivative

Given a function $y = f(x)$, the derivative, $f'(x)$ is a formula for the slope of the tangent line. Conceptually, the derivative represents the instantaneous rate of change of the function. Other notations for the derivative include $y'$ and $\frac{dy}{dx}$.



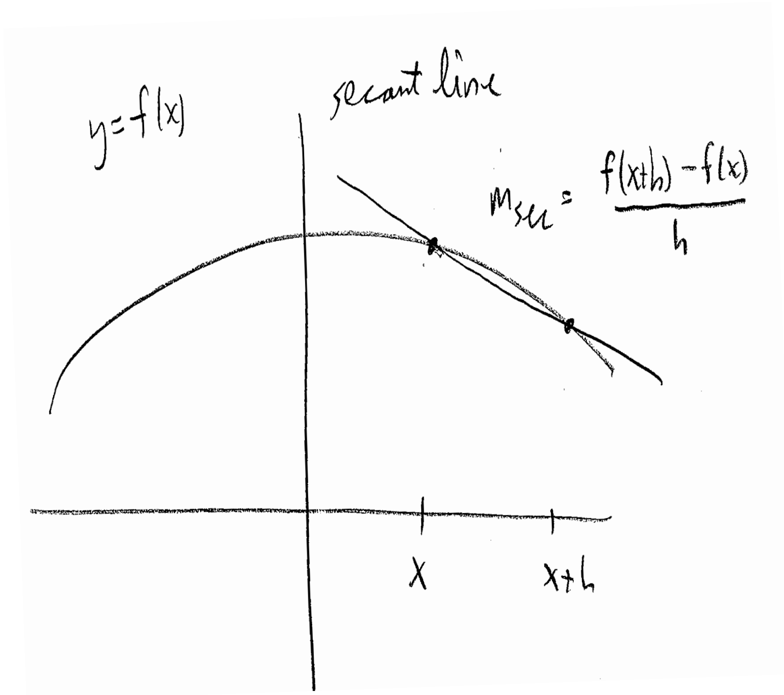

To begin the discussion, we recall the formula for the slope of a line between two points, $(x_1, y_1)$ and $(x_2, y_2)$:

$$m = \frac{\text{rise}}{\text{run}} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}.$$

To find the slope of the tangent line to the graph of $y = f(x)$ at the point $(x, f(x))$, we first compute the slope of the secant line connecting the point $(x, f(x))$ with the nearby point $(x + h, f(x + h))$:

$$m_{\text{sec}} = \frac{f(x+h) - f(x)}{(x+h) - x} = \frac{f(x+h) - f(x)}{h}.$$

This quantity is frequently referred to as the **difference quotient**.




To find the slope of the tangent line, we need to move the point $(x+h, f(x+h))$ closer and closer to the point $(x, f(x))$ without ever reaching it. This requires the use of a limit. To make $x+h$ approach $x$ we must make $h$ approach 0.

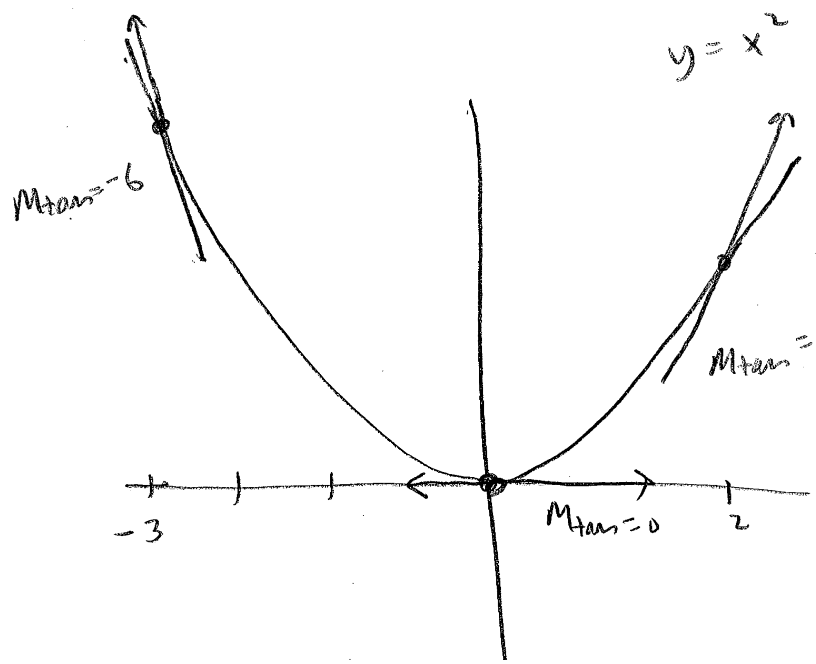
This gives us the **definition of the derivative**:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

Let's compute the derivative of $x^2$ and interpret some of its values. Using the formula for $f'(x)$, we have

$$\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} \\
&= \lim_{h \to 0} \frac{(x^2 + 2xh + h^2) - x^2}{h} \\
&= \lim_{h \to 0} \frac{(2xh + h^2)}{h} \\
&= \lim_{h \to 0} \frac{h(2x + h)}{h} \\
&= \lim_{h \to 0} (2x + h) \\
&= 2x.
\end{aligned}$$

Thus the derivative of $x^2$ is $2x$ and this tells use the slope of the tangent line at a given $x$-value. For example, if $x = -3$, then the derivative is $f'(-3) = 2(-3) = -6$. This means that the slope of the tangent line at the point $(-3, 9)$ is $-6$. At the point $(0, 0)$, the slope of the tangent line is $f'(0) = 2(0) = 0$, and at the point $(2, 4)$ the slope is $f'(2) = 2(2) = 4$.




We now compute the derivative of several different functions.

### Examples Using the Definition of the Derivative

**DD 1.** Let $f(x) = x^2 - 5x + 7$. To find $f'(x)$ using the definition of derivative, we will need to compute $f(x + h)$. To do this, we substitute the expression $(x + h)$ into the formula for $f(x)$ in the place of $x$. In this example, we get,

$$\begin{aligned} f(x+h) &= (x+h)^2 - 5(x+h) + 7 \\ &= x^2 + 2xh + h^2 - 5x - 5h + 7. \end{aligned}$$

Now we use the definition:

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{(x^2+2xh+h^2-5x-5h+7)-(x^2-5x+7)}{h} \\
&= \lim_{h\to 0} \frac{2xh+h^2-5h}{h} \\
&= \lim_{h\to 0} \frac{h(2x+h-5)}{h} \\
&= \lim_{h\to 0} (2x+h-5) \\
&= 2x-5.
\end{aligned}
$$

To recap, the derivative of $x^2-5x+7$ is $2x-5$.

**DD 2.** If $f(x)=x^3$ then in order to find $f'(x)$ we first compute $f(x+h)$.

We have

$$
\begin{aligned}
f(x+h) &= (x+h)^3 \\
&= (x+h)(x+h)^2 \\
&= (x+h)(x^2+2xh+h^2) \\
&= x^3+2x^2h+xh^2+x^2h+2xh^2+h^3 \\
&= x^3+3x^2h+3xh^2+h^3.
\end{aligned}
$$

Now we use the definition:

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{(x^3+3x^2h+3xh^2+h^3)-x^3}{h} \\
&= \lim_{h\to 0} \frac{3x^2h+3xh^2+h^3}{h} \\
&= \lim_{h\to 0} \frac{h(3x^2+3xh+h^2)}{h} \\
&= \lim_{h\to 0} (3x^2+3xh+h^2) \\
&= 3x^2.
\end{aligned}
$$

To recap, the derivative of $x^3$ is $3x^2$.

**DD 3.** If $f(x) = \sqrt{x}$ then

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} \\
&= \lim_{h\to 0} \frac{(x+h) - (x)}{h(\sqrt{x+h} + \sqrt{x})} \\
&= \lim_{h\to 0} \frac{h}{h(\sqrt{x+h} + \sqrt{x})} \\
&= \lim_{h\to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} \\
&= \frac{1}{2\sqrt{x}}.
\end{aligned}$$

To recap, the derivative of $\sqrt{x}$ is $\dfrac{1}{2\sqrt{x}}$.

**DD 4.** If $f(x) = \sqrt{2x+1}$ then

$$\begin{aligned}
f(x+h) &= \sqrt{2(x+h)+1} \\
&= \sqrt{2x+2h+1}.
\end{aligned}$$

Now for the derivative:

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{2x+2h+1}-\sqrt{2x+1}}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{2x+2h+1}-\sqrt{2x+1}}{h} \cdot \frac{\sqrt{2x+2h+1}+\sqrt{2x+1}}{\sqrt{2x+2h+1}+\sqrt{2x+1}} \\
&= \lim_{h\to 0} \frac{(2x+2h+1)-(2x+1)}{h(\sqrt{2x+2h+1}+\sqrt{2x+1})} \\
&= \lim_{h\to 0} \frac{2h}{h\sqrt{2x+2h+1}+\sqrt{2x+1}} \\
&= \lim_{h\to 0} \frac{2}{\sqrt{2x+2h+1}+\sqrt{2x+1}} \\
&= \frac{2}{\sqrt{2x+1}+\sqrt{2x+1}} \\
&= \frac{2}{2\sqrt{2x+1}} \\
&= \frac{1}{\sqrt{2x+1}}.
\end{aligned}$$

To recap, the derivative of $\sqrt{2x+1}$ is $\dfrac{1}{\sqrt{2x+1}}$.

**DD 5.** If $f(x) = \dfrac{1}{x}$ then

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{\frac{1}{x+h}-\frac{1}{x}}{h} \\
&= \lim_{h\to 0} \frac{\frac{(x)-(x+h)}{x(x+h)}}{h} \\
&= \lim_{h\to 0} \frac{-h}{x(x+h)} \cdot \frac{1}{h} \\
&= \lim_{h\to 0} \frac{-1}{x(x+h)} \\
&= -\frac{1}{x^2}.
\end{aligned}$$

To recap, the derivative of $\dfrac{1}{x}$ is $-\dfrac{1}{x^2}$.

**DD 6.** If $f(x) = \dfrac{2}{x+3}$ then

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{\frac{2}{x+h+3} - \frac{2}{x+3}}{h} \\
&= \lim_{h\to 0} \frac{\frac{2(x+3)-2(x+h+3)}{(x+h+3)(x+3)}}{h} \\
&= \lim_{h\to 0} \frac{\frac{2x+6-2x-2h-6}{(x+h+3)(x+3)}}{h} \\
&= \lim_{h\to 0} \frac{-2h}{(x+h+3)(x+3)} \cdot \frac{1}{h} \\
&= \lim_{h\to 0} \frac{-2}{(x+h+3)(x+3)} \\
&= -\frac{2}{(x+3)^2}.
\end{aligned}$$

To recap, the derivative of $\dfrac{2}{x+3}$ is $-\dfrac{2}{(x+3)^2}$.

The next two examples use the special limits:

$$\lim_{h\to 0} \frac{\cos(h)-1}{h} = 0 \quad \text{and} \quad \lim_{h\to 0} \frac{\sin(h)}{h} = 1.$$

**DD 7.** If $f(x) = \sin(x)$ then

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{\sin(x+h)-\sin(x)}{h} \\
&= \lim_{h\to 0} \frac{\sin(x)\cos(h)+\cos(x)\sin(h)-\sin(x)}{h} \\
&= \lim_{h\to 0} \frac{\sin(x)(\cos(h)-1)+\cos(x)\sin(h)}{h} \\
&= \lim_{h\to 0} \frac{\sin(x)(\cos(h)-1)}{h} + \frac{\cos(x)\sin(h)}{h} \\
&= \sin(x)\cdot 0 + \cos(x)\cdot 1 \\
&= \cos(x).
\end{aligned}$$

To recap, the derivative of $\sin(x)$ is $\cos(x)$.

**DD 8.** If $f(x) = \cos(x)$ then

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{\cos(x+h) - \cos(x)}{h} \\
&= \lim_{h\to 0} \frac{\cos(x)\cos(h) - \sin(x)\sin(h) - \cos(x)}{h} \\
&= \lim_{h\to 0} \frac{\cos(x)[\cos(h) - 1] - \sin(x)\sin(h)}{h} \\
&= \lim_{h\to 0} \frac{\cos(x)[\cos(h) - 1]}{h} - \frac{\sin(x)\sin(h)}{h} \\
&= \cos(x) \cdot 0 - \sin(x) \cdot 1 \\
&= -\sin(x).
\end{aligned}
$$

To recap, the derivative of $\cos(x)$ is $-\sin(x)$.

The next example uses the special limit

$$\lim_{h\to 0} \frac{e^h - 1}{h} = 1.$$

**DD 9.** If $f(x) = e^x$ then

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{e^{x+h} - e^x}{h} \\
&= \lim_{h\to 0} \frac{e^x e^h - e^x}{h} \\
&= \lim_{h\to 0} \frac{e^x(e^h - 1)}{h} \\
&= e^x \cdot 1 = e^x.
\end{aligned}
$$

To recap, the derivative of $e^x$ is $e^x$.

The next example uses the special limit

$$\lim_{k\to 0} \frac{\ln(1+k)}{k} = 1.$$

**DD 10.** If $f(x) = \ln(x)$ then

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{\ln(x+h) - \ln(x)}{h} \\
&= \lim_{h\to 0} \frac{\ln(\frac{x+h}{x})}{h} \\
&= \lim_{h\to 0} \frac{\ln(1+\frac{h}{x})}{h} \\
&= \lim_{k\to 0} \frac{\ln(1+k)}{kx} \\
&= \frac{1}{x}.
\end{aligned}$$

Note that we made the substitution $k = \frac{h}{x}$ so that $h = kx$ and also note that $h \to 0$ is equivalent to $k \to 0$. To recap, the derivative of $\ln(x)$ is $\frac{1}{x}$.