0.1 The Fundamental Theorem of Calculus

The expression $\int_a^b f(x)\,dx$ is called a definite integral. The numbers a and b on the integral sign are called the **endpoints of integration** and the function $f(x)$ is called the **integrand**. The Fundamental Theorem of Calculus tells us how to compute a definite integral.

Fundamental Theorem of Calculus

If $f(x)$ is continuous on the interval $[a,b]$ and F is an anti-derivative of f, then

$$\int_a^b f(x)\ dx = F(b) - F(a).$$

The process of calculating the numerical value of a definite integral is performed in two main steps: first, find the function F and second, plug in the endpoints of integration. To symbolize this we use a vertical evaluation bar:

$$\int_a^b f(x)\ dx = F(x)\Big|_a^b = F(b) - F(a)$$

Examples of the Fundamental Theorem of Calculus

FT 1. $\displaystyle\int_0^{\pi/2} \cos(x)\ dx = \sin(x)\Big|_0^{\pi/2} = \sin(\tfrac{\pi}{2}) - \sin(0) = 1 - 0 = 1.$

FT 2. $\displaystyle\int_0^{\ln(3)} e^x\ dx = e^x\Big|_0^{\ln(3)} = e^{\ln(3)} - e^0 = 3 - 1 = 2.$

FT 3. $\displaystyle\int_1^2 x^2\ dx = \frac{x^3}{3}\Big|_1^2 = \tfrac{2^3}{3} - \tfrac{1^3}{3} = \tfrac{7}{3}.$

FT 4. $\displaystyle\int_1^e \frac{1}{x}\ dx = \ln|x|\Big|_1^e = \ln(e) - \ln(1) = 1 - 0 = 1.$

FT 5. $\displaystyle\int_0^2 (2x+1)\ dx = (x^2+x)\Big|_0^2 = (2^2+2) - (0^2+0) = 6.$

FT 6. $\displaystyle\int_0^{\pi/4} \sec^2(x)\ dx = \tan(x)\Big|_0^{\pi/4} = \tan(\tfrac{\pi}{4}) - \tan(0)$
$= 1 - 0 = 1.$

FT 7. $\displaystyle\int_0^1 \frac{1}{1+x^2}\ dx = \tan^{-1}(x)\Big|_0^1 = \tan^{-1}(1) - \tan^{-1}(0)$
$= \tfrac{\pi}{4} - 0 = \tfrac{\pi}{4}.$

FT 8.
$$\begin{aligned}\int_{-1}^{1} (3x+5)\ dx &= \left(\tfrac{3}{2}x^2 + 5x\right)\Big|_{-1}^{1} \\ &= (\tfrac{3}{2}(1^2) + 5(1)) - (\tfrac{3}{2}(-1)^2 + 5(-1)) \\ &= (\tfrac{3}{2} + 5) - (\tfrac{3}{2} - 5) \\ &= 10.\end{aligned}$$