0.1 Critical Numbers

In this section we will find the **critical numbers** of a given function. Critical numbers come in two main types and the idea of the definition is to consider the possibilities at a local extreme. Here is the definition of critical number.

Definition of Critical Number

A critical number for the function $f(x)$ is a number x_0 in the *domain* of the function $f(x)$ such that either

$$f'(x_0) = 0 \text{ (type 1)}$$

or

$$f'(x_0) \text{ is undefined (type 2).}$$

In other words, at a critical number x_0 we have $f(x_0)$ is defined and either $f'(x_0) = 0$ or $f(x)$ is not differentiable at x_0 .

An example of the first type is $x_0 = 0$ for the function $f(x) = x^2$ since $f(0)$ is defined (it equals 0) and $f'(0) = 0$ since $f'(x) = 2x$.

An example of the second type is $x_0 = 0$ for the function $f(x) = |x|$ since $f(0)$ is defined (it equals 0) and $f'(0)$ is undefined since $f(x) = |x|$ has a corner point at $x = 0$ and hence it is not differentiable there.

Examples of Finding Critical Numbers

CN 1. Find the critical numbers of the function

$$f(x) = \frac{x^3}{3} - \frac{x^2}{2} - 6x - 3.$$

Solution: We need to compute $f'(x)$. We have

$$f'(x) = x^2 - x - 6.$$

Noting that $f'(x)$ is defined for all values of x, there are no type 2 critical numbers. To find the type 1 critical numbers, we solve the equation

$$f'(x) = 0.$$

Geometrically, these are the points where the graph of $f(x)$ has horizontal tangent lines. We get

$$x^2 - x - 6 = 0$$

$$(x+2)(x-3) = 0$$

$$x = -2 \text{ or } x = 3.$$

Hence $f(x) = \frac{x^3}{3} - \frac{x^2}{2} - 6x - 3$ has two critical numbers, -2 and 3, and they are both type 1.

CN 2. Find the critical numbers of the function

$$f(x) = x^2 e^{3x}.$$

We need to compute $f'(x)$ using the product and chain rules. We have

$$f'(x) = (3x^2 + 2x)e^{3x} \text{ (verify)}.$$

Noting that $f'(x)$ is defined for all values of x, there are no type 2 critical numbers. To find the type 1 critical numbers, we solve the equation

$$f'(x) = 0.$$

Geometrically, these are the points where the graph of $f(x)$ has horizontal tangent lines. We get

$$(3x^2 + 2x)e^{3x} = 0$$

$$x(3x+2)e^{3x} = 0$$

$$x = 0 \text{ or } x = -2/3.$$

Note that the equation $e^{3x} = 0$ has no solutions since an exponential function is always positive.

Hence $f(x) = x^2 e^{3x}$ has two critical numbers, $-2/3$ and 0, and they are both type 1.

CN 3. Find the critical numbers of the function

$$f(x) = \frac{x}{x^2+1}.$$

We need to compute $f'(x)$ using the quotient rule. We have

$$f'(x) = \frac{1-x^2}{(x^2+1)^2} \text{ (verify).}$$

Noting that $f'(x)$ is defined for all values of x (since the denominator is never equal to 0), there are no type 2 critical numbers. To find the type 1 critical numbers, we solve the equation

$$f'(x) = 0.$$

Geometrically, these are the points where the graph of $f(x)$ has horizontal tangent lines. We get

$$\frac{1-x^2}{(x^2+1)^2} = 0$$

$$1 - x^2 = 0$$

$$x = \pm 1.$$

Hence $f(x) = \dfrac{x}{x^2+1}$ has two critical numbers, -1 and 1, and they are both type 1.

CN 4. Find the critical numbers of the function

$$f(x) = \sqrt[3]{x}.$$

We need to compute $f'(x)$. We have

$$f'(x) = \frac{1}{3\sqrt{x}^3} \text{ (verify)}.$$

In this case, $f'(0)$ is undefined (division by zero). Hence $x = 0$ is a critical number **if** $f(0)$ is defined. We can easily check this: $f(0) = \sqrt[3]{0} = 0$, so it is defined and now we can conclude that $x = 0$ is a type 2 critical number. To find the type 1 critical numbers, we solve the equation

$$f'(x) = 0.$$

Geometrically, these are the points where the graph of $f(x)$ has horizontal tangent lines. We get

$$\frac{1}{3\sqrt{x}^3} = 0$$

$$1 = 0.$$

So there are no solutions. The function has no type 1 critical numbers.

Hence $f(x) = \sqrt[3]{x}$ has only one critical number, 0, and it type 2, where the function is not differentiable. Geometrically, the function $f(x) = \sqrt[3]{x}$ has a vertical tangent line at the critical number $x = 0$.

CN 5. Find the critical numbers of the function

$$f(x) = \frac{1}{x}.$$

We need to compute $f'(x)$. We have

$$f'(x) = -\frac{1}{x^2} \text{ (verify)}.$$

In this case, $f'(x)$ is undefined at $x = 0$ (division by zero). Hence, **if** $f(0)$ is defined then $x = 0$ would be a type 2 critical number. However, we can easily see that $f(0) = \frac{1}{0}$, is undefined, so that $x = 0$ is a not in the domain of $f(x)$ and hence it is **not** a critical number. To find the type 1 critical numbers, we solve the equation

$$f'(x) = 0.$$

Geometrically, these are the points where the graph of $f(x)$ has horizontal tangent lines. We get

$$-\frac{1}{x^2} = 0$$

$$1 = 0.$$

So there are no solutions. The function has no type 1 critical numbers.

Hence $f(x) = \frac{1}{x}$ has no critical numbers.