0.1 Continuity

One of the tools we used in the last section for evaluating limits analytically was plugging in. For example, to compute

$$\lim_{x\to 5} x^2$$

we would simply plug in $x = 5$ and get the correct answer, 25. In most of the examples in the last section, plugging in did not give an acceptable answer at first, so more work had to be done before eventually plugging in. If plugging does work at the outset, then the function is called continuous.

Definition of Continuity

We say that the function $f(x)$ is **continuous** at $x = a$ if the following limit holds:

$$\lim_{x\to a} f(x) = f(a).$$

That is to say that the function $f(x)$ is continuous at $x = a$ if the limit can be computed by plugging in. Thus, from the limit given earlier,

$$\lim_{x\to 5} x^2 = 5^2,$$

we can say that the function $f(x) = x^2$ is continuous at $x = 5$.

As an example, the function $f(x) = \tan(x)$ is not continuous at $x = \pi/2$ because $\tan(\pi/2)$ is undefined and hence

$$\lim_{x\to \frac{\pi}{2}} \tan(x) \neq \tan(\pi/2).$$

If a function is continuous at each point in an interval, I, then we say that $f(x)$ is **continuous on** I.

A function $f(x)$ is called **continuous from the left** at $x = a$ if

$$\lim_{x\to a^-} f(x) = f(a),$$

and it is called **continuous from the right** at $x = a$ if

$$\lim_{x \to a^+} f(x) = f(a).$$

Note that if $f(x)$ is continuous at $x = a$ then it is both continuous from the left and continuous from the right at $x = a$.

Types of Discontinuities

We say that $f(x)$ has a **discontinuity** at $x = a$ if

$$\lim_{x \to a} f(x) \neq f(a).$$

Depending on the reason for these two quantities being unequal, we will get a different type of discontinuity. There are four types of discontinuities discussed below.

First, the discontinuity is called **removable** if

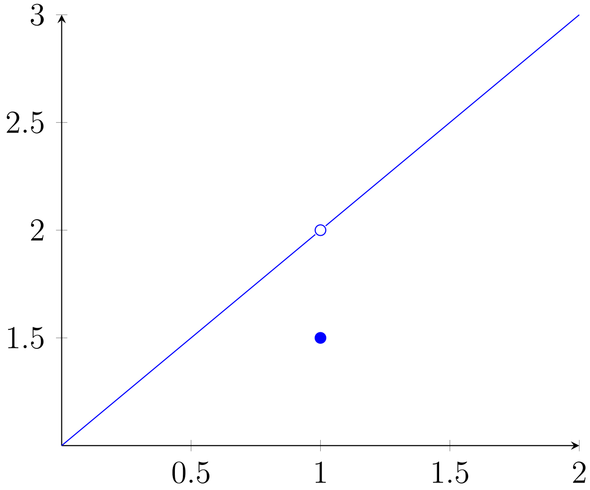
$$\lim_{x \to a} f(x) \ \ \textbf{exists},$$

but is not equal to $f(a)$.

In other words, the limit is equal to a number but that number is not $f(a)$. This might be because $f(a)$ is undefined or it could be that, even though $f(a)$ is defined, it just does not equal the limit. Such a discontinuity is called removable because if we were to define the function appropriately at $x = a$ it would become continuous, i.e., the discontinuity would have been removed.

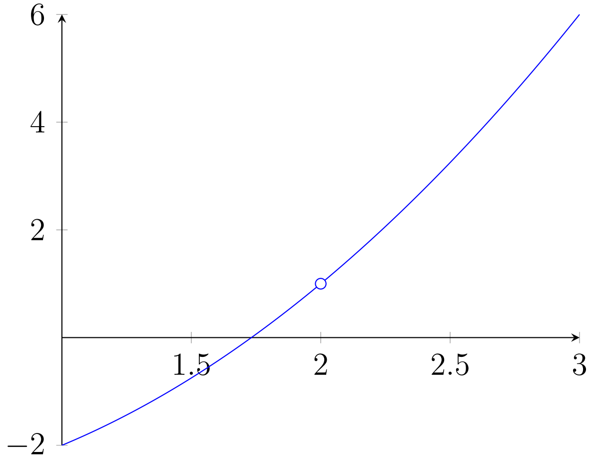
Here are two examples of graphs of functions that have removable discontinuities:

A Removable Discontinuity at $x = 1$



The two-sided limit and the function value are unequal.

A Removable Discontinuity at $x = 2$



The function value is undefined.

The second type of discontinuity is called a **jump**. This happens when the limit

$$\lim_{x \to a} f(x) \text{ does not exist}$$

for the following reason:

$$\lim_{x \to a^-} f(x) \text{ exists, and}$$

$$\lim_{x \to a^+} f(x) \text{ exists, but}$$

$$\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x).$$

In other words, the one sided limits both exist as finite numbers but they are not equal to each other, and therefore the two sided limit does not exist.

As an example, the function

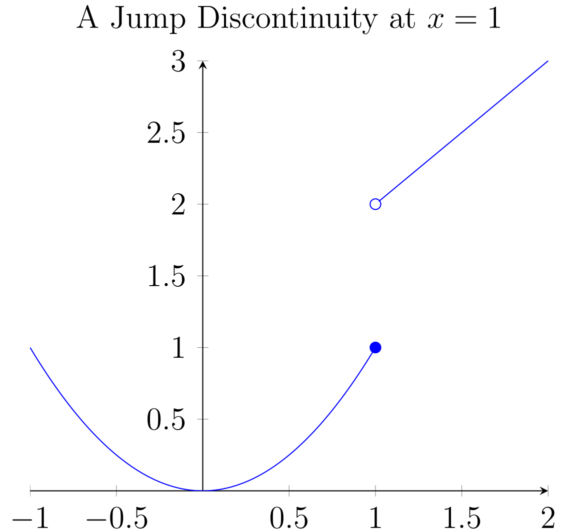
$$f(x) = \frac{|x|}{x}$$

has a jump discontinuity at $x = 0$ because

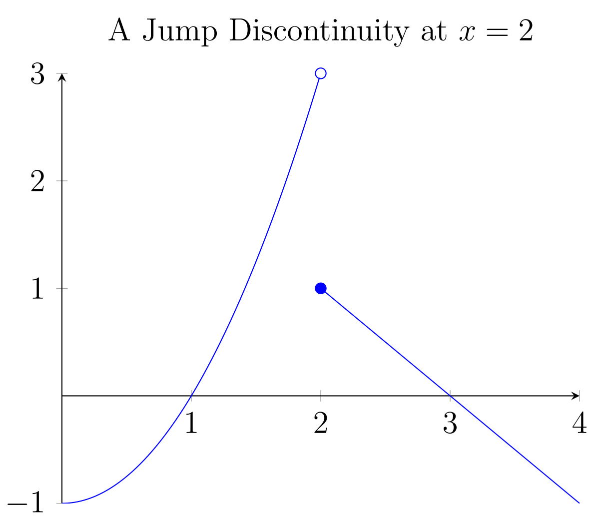
$$\lim_{x \to 0^-} \frac{|x|}{x} = -1 \text{ but}$$

$$\lim_{x \to 0^+} \frac{|x|}{x} = 1.$$

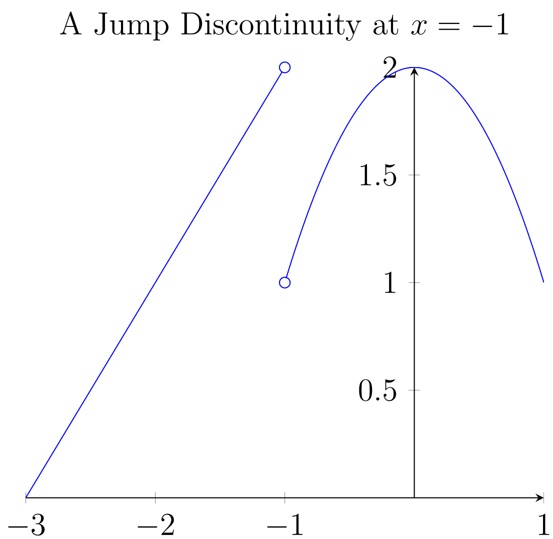
Here are three examples of graphs of functions that have jump discontinuities:



The one-sided limits exist but are unequal.



The one-sided limits exist but are unequal.



The one-sided limits exist but are unequal.

The third type of discontinuity is called **infinite**. This occurs if the function has a vertical asymptote at $x = a$. In terms of limits, $f(x)$ has an infinite discontinuity at $x = a$ if

$$\lim_{x \to a^-} f(x) = \pm\infty, \text{ or}$$

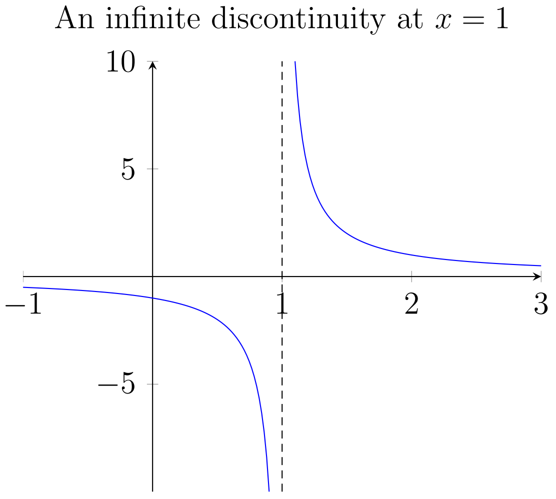
$$\lim_{x \to a^+} f(x) = \pm\infty.$$

As an example, the function $f(x) = \tan(x)$ has an infinite discontinuity at $x = \frac{\pi}{2}$ since

$$\lim_{x \to \frac{\pi}{2}^-} \tan(x) = \infty \text{ and}$$

$$\lim_{x \to \frac{\pi}{2}^+} \tan(x) = -\infty.$$

Here is an example of the graph of a function that has an infinite discontinuity.

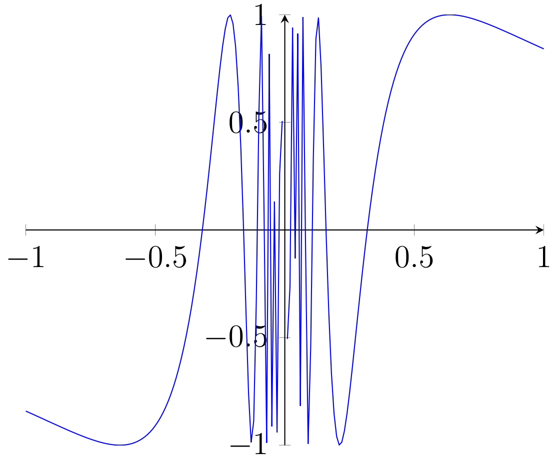


Fourth and finally, if the discontinuity is not one of the first three types, it is called an **essential** discontinuity. As an example, the function $f(x) = \sin(\frac{1}{x})$ has an essential discontinuity at $x = 0$. Both one-sided limits do not exist due to oscillation of the function, but the function does not have a vertical asymptote since

$$-1 \leq \sin\left(\frac{1}{x}\right) \leq 1$$

for all values of x except $x = 0$, where the function is undefined. Here is an example of the graph of a function that has an essential discontinuity:

An Essential Discontinuity at $x = 0$



Familiar Functions

Consider the function $f(x) = x^2$ and any number a. We can compute the limit

$$\lim_{x \to a} x^2 = a^2$$

by plugging in. Therefore, we can say that $f(x) = x^2$ is continuous for all real numbers. We can also say that $f(x) = x^2$ is continuous on the interval $(-\infty, \infty)$. Actually, this is true for all polynomials, including constant functions. If $p(x)$ is a polynomial, then $p(x)$ is continuous on the interval $(-\infty, \infty)$. There are some other familiar functions which are also continuous on the interval $(-\infty, \infty)$. These are:

$$f(x) = \sin(x), \cos(x), e^x, \tan^{-1}(x), \sqrt[3]{x} \text{ and } |x|.$$

The function $f(x) = \ln(x)$ is only defined for x in the interval $(0, \infty)$ and it is continuous on this interval.
The function $f(x) = \tan(x)$ has vertical asymptotes at odd

multiples of $\frac{\pi}{2}$. It is continuous between these vertical asymptotes, so, for example, $f(x) = \tan(x)$ is continuous on the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$. The function $f(x) = \sec(x)$ is similar to $\tan(x)$ in that it has vertical asymptotes at odd multiples of $\frac{\pi}{2}$, and it is continuous on the intervals between them. The functions $\cot(x)$ and $\csc(x)$ have vertical asymptotes at multiples of π and like $\tan(x)$ and $\sec(x)$, they are continuous between their asymptotes. For example, the function $f(x) = \cot(x)$ is continuous on the interval $(0, \pi)$.

The function $f(x) = \sqrt{x}$ is only defined for $x \geq 0$ and it is continuous for all of these values of x. In other words, $f(x) = \sqrt{x}$ is continuous on the interval $[0, \infty)$. To say that the function is continuous at the left endpoint of this interval $(x = 0)$ it is sufficient that the function is right continuous at this point. And it is indeed true that

$$\lim_{x \to 0^+} \sqrt{x} = \sqrt{0} = 0.$$

In general, a root function, $f(x) = \sqrt[n]{x}$ is continuous on the interval $[0, \infty)$ if n is even and it is continuous on the interval $(-\infty, \infty)$ if n is odd.

The last type of familiar function that we will discuss here is the rational function. A rational function is a ratio of polynomials,

$$f(x) = \frac{p_1(x)}{p_2(x)},$$

where $p_1(x)$ and $p_2(x)$ are both polynomials and the degree of $p_2(x)$ is at least 1. Such a function is continuous for all values of x such that $p_2(x) \neq 0$. For example, the function

$$f(x) = \frac{x}{x^2 + 1}$$

is continuous on the interval $(-\infty, \infty)$ since $x^2+1 \neq 0$ for any x. On the other hand, the function

$$f(x) = \frac{x}{x^2-1}$$

is continuous on the intervals $(-\infty, -1), (-1, 1)$ and $(1, \infty)$ since $x^2 - 1 = 0$ when $x = \pm 1$.

Properties of Continuity

Continuous functions combine nicely with respect to the operations addition, subtraction, multiplication, division and composition. Specifically, if $f(x)$ and $g(x)$ are both continuous at $x = a$, then so are

$$f(x) + g(x),$$

$$f(x) - g(x) \text{ and}$$

$$f(x) \cdot g(x).$$

Furthermore, if $g(a) \neq 0$ then

$$\frac{f(x)}{g(x)}$$

is also continuous at $x = a$. In words, we say that the sum, difference, product and quotient of continuous functions is continuous (with the understanding that $g(a) \neq 0$ in the case of the quotient.) Things are slightly more complicated for the composition. If $g(x)$ is continuous at $x = a$ and $f(x)$ is continuous at $x = g(a)$ then then composition $f(g(x))$ is continuous at $x = a$. This situation is different from the four basic operations because in composition, when plugging in $x = a$, we plug a into $g(x)$ and then plug $g(a)$ into $f(x)$, whereas for the first four basic operations we plug $x = a$ into both $f(x)$ and $g(x)$.

Piecewise Function

In this section we will examine continuous and discontinuous piecewise defined functions.

PW 1. The function given by

$$f(x) = \begin{cases} x - 1 & \text{for } x \leq 2 \\ 3 & \text{for } x > 2 \end{cases}$$

has a discontinuity at $x = 2$. This is because

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} x - 1 = 2 - 1 = 1$$

but

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} 3 = 3.$$

Since the one-sided limits are different, the two-sided limit

$$\lim_{x \to 2} f(x)$$

does not exist and hence

$$\lim_{x \to 2} f(x) \neq f(2).$$

This is an example of a jump discontinuity, since the two sided limit does not exist, but the one-sided limits are finite.

PW 2. The function given by

$$f(x) = \begin{cases} x - 1 & \text{for } x \leq 2 \\ x^2 - 3 & \text{for } x > 2 \end{cases}$$

is continuous at $x = 2$. This is because

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} x - 1 = 2 - 1 = 1$$

and

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} x^2 - 3 = 2^2 - 3 = 1.$$

Since the one-sided limits are equal, the two-sided limit

$$\lim_{x\to 2} f(x) = 1$$

as well. Finally, $f(2) = 2 - 1 = 1$ and so

$$\lim_{x\to 2} f(x) = f(2)$$

and the function is continuous at $x = 2$.

PW 3. The function given by

$$f(x) = \begin{cases} x - 1 & \text{for } x < 2 \\ 3 & \text{for } x = 2 \\ x^2 - 3 & \text{for } x > 2 \end{cases}$$

has a discontinuity at $x = 2$. This is because

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} x - 1 = 2 - 1 = 1$$

and

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} x^2 - 3 = 2^2 - 3 = 1.$$

Since the one-sided limits are equal, the two-sided limit

$$\lim_{x\to 2} f(x) = 1$$

as well. However, $f(2) = 3$ and so

$$\lim_{x\to 2} f(x) \neq f(2)$$

and the function is discontinuous at $x = 2$. This is an example of a removable discontinuity, since

$$\lim_{x\to 2} f(x)$$

is a finite number.

PW 4. Consider the function

$$f(x) = \begin{cases} kx - 5 & \text{for } x \leq 2 \\ x^2 - k & \text{for } x > 2 \end{cases}$$

If we choose the parameter k correctly, we the function will be continuous at $x = 2$. To determine the correct value of k we compare the one-sided limits:

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} kx - 5 = 2k - 5$$

and

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} x^2 - k = 2^2 - k = 4 - k.$$

In order for the function to be continuous, these need to be the same. Hence, we set them equal to each other and we solve for k:

$$2k - 5 = 4 - k$$

$$3k = 9$$

$$k = 3.$$

With this choice of k, the function becomes

$$f(x) = \begin{cases} 3x - 5 & \text{for } x \leq 2 \\ x^2 - 3 & \text{for } x > 2 \end{cases}$$

and it is continuous at $x = 2$ because

$$\lim_{x \to 2} f(x) = f(2) = 1.$$

PW 5. Consider the function

$$f(x) = \begin{cases} kx^2 - 8 & \text{for } x \leq 2 \\ x^2 - kx & \text{for } x > 2 \end{cases}$$

If we choose the parameter k correctly, the function will be continuous at $x = 2$. To determine the correct value of k we compare the one-sided limits:

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} kx^2 - 8 = 4k - 8$$

and

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} x^2 - kx = 4 - 2k.$$

In order for the function to be continuous, these need to be the same. Hence, we set them equal to each other and we solve for k:

$$4k - 8 = 4 - 2k$$

$$6k = 12$$

$$k = 2.$$

With this choice of k, the function becomes

$$f(x) = \begin{cases} 2x^2 - 8 & \text{for } x \leq 2 \\ x^2 - 2x & \text{for } x > 2 \end{cases}$$

and it is continuous at $x = 2$ because

$$\lim_{x\to 2} f(x) = f(2) = 0.$$