Math 320 Projects
(courtesy of Bill Briggs, Tessa Weinstein, and Jim Sollazo)

Introduction and Guidelines

This is a collection of assorted projects that are supported by the material that we will study in Math 320 this semester. You must complete three (3) projects during the semester. You may collaborate on the projects, but the final write-up that you submit must be entirely your own work. Your write-ups should be presented neatly with supporting figures, graphs or tables, complete solutions, and discussion of results. Write-ups should be detailed enough that a fellow student could understand your solution. You are welcome to ask me for help or clarification at any time.

Do not base your choice of projects on their apparent length! The projects are designed to require roughly the same amount of time and effort if you are prepared. Some are quite applied and others are more theoretical in nature. The projects are listed, more or less, in the order in which we will study the relevant material in class. You should be able to get started almost immediately. Do not postpone doing the projects until the week that they are due.

Project 1 - Draining Reservoirs

Imagine a large water reservoir which loses water due to evaporation. In all that follows, we will let \( h(t), S(t) \) and \( V(t) \) denote the depth, the surface area and the volume of the water in the reservoir, respectively, at time \( t \). We will always assume that the rate of change of the water volume is proportional to the area of the exposed water in the reservoir; that is, \( V'(t) = -\alpha S(t) \), where \( \alpha > 0 \) is a constant (notice the minus sign). In all that follows, we will assume that \( \alpha = 0.05 \) meters/day. Note: anytime the instructions below say verify you can interpret this as show or derive.

1. First consider a reservoir that has the shape of rectangular prism (parallelepiped) with a constant horizontal cross-sectional area of 200 square meters and a depth of 10 meters.

   (a) Verify that the units of \( \alpha \) are consistent.
   (b) Assuming that the reservoir is filled at \( t = 0 \), what is the initial volume of the water?
   (c) Noting that in this case the surface area of the reservoir is constant, solve the ODE that governs the change in water volume. Use the initial condition to express the volume as a function of time.
   (d) At what time (after how many days) will the reservoir be empty, assuming the evaporation rate remains constant?
2. Now assume that the reservoir is shaped like an inverted frustum of a pyramid: an upside-down square-based pyramid whose top has been sliced off, see Figure 2. The horizontal cross-sections of the reservoir are always squares which decrease in area from 225 square meters at the top of the reservoir to 100 square meters at the bottom. The depth of the reservoir is 10 meters.

![Reservoir](image)

Figure 1: Reservoir

(a) The volume of a pyramid is given by \( V = \frac{1}{3}Ah \) where \( A \) is the area of the (square) base and \( h \) is the height (see right figure below). Verify that the volume of water in this reservoir when it is full is \( V_0 = 1583.33 \) cubic meters. (Consider the entire pyramid.)

(b) Again letting \( h \) denote the water depth in the reservoir, verify that the surface of the water in the reservoir is a square whose sides have length given by \( l(h) = 10 + h/2 \). (Check that \( l(0) = 10 \) and \( l(10) = 15 \).)

(c) Verify that the surface area of the water is \( S(h) = (l(h))^2 \). (Check that \( S(0) = 100 \) and \( S(10) = 225 \).)

(d) Verify that the volume of water in the reservoir is given by \( V(h) = \frac{1}{3}S(h)(20 + h) - \overline{v} \), where \( \overline{v} = 666.67 \) cubic meters. (Check that \( V(0) = 0 \) and \( V(10) = V_0 \).)

A derivation is required.

(e) Important step: In order to use the ODE \( V'(t) = -\alpha S(t) \), we must relate the surface area directly to the volume. Show that \( S = \frac{1}{4}(12(V + \overline{v}))^{2/3} \).

(f) It now follows that the governing ODE for the volume is \( V'(t) = -a(V + \overline{v})^{2/3} \) where \( a = 3\alpha/(12)^{1/3} \).

(g) Solve this ODE (using the initial condition \( V(0) = V_0 \)) and graph the solution.

(h) At what time (after how many days) will the reservoir be empty, assuming the evaporation rate remains constant?

Project 2 - Free Fall and Terminal Velocity
We learned in class (see also section 3.4 of the text) that an object in free fall in a gravitational field is governed by the ODE

\[ m \frac{dv}{dt} = mg + F_s, \]  

(1)

where \( m \) is the mass of the object, \( g = 9.8 \) meters/sec\(^2\) is the acceleration of gravity, \( v(t) \) is the velocity of the object \( t \) seconds after it is released, and \( F_s \) denotes external forces acting on the object. In all that follows, assume that \( v(0) = 0 \). In this problem, since we will investigate free fall and terminal velocity, let's choose the positive direction for velocity and position as downward, in the same direction as \( g \); therefore, the coefficient of \( mg \) in the ODE is \(+1\), not \(-1\).

1. If there are no external forces acting on the object, then its velocity increases without bound (until the object collides with something). This is unrealistic for motion in the earth’s atmosphere, since air resistance is a significant effect. Therefore, assume that air resistance is present and is described by \( F_s = -kv \) (\( k \) is a constant and the minus sign indicates that the air resistance opposes the motion). Show (include all steps) that the solution of this ODE is

\[ v(t) = \frac{mg}{k}(1 - e^{-kt/m}). \]  

(2)

2. What is the terminal velocity \( v_T = \lim_{t \to \infty} \) of a 100-kilogram object (a small linebacker or a large flower pot) subject to air resistance described by \( k = 5 \) kg/sec?

3. Find the function that describes the position \( x(t) \) of the object for all \( t \geq 0 \) assuming that \( x = 0 \) corresponds to the position at which the object is dropped.

4. Make sketches of \( v(t) \) and \( x(t) \), use computer.

5. Now assume that \( F_s = -kv^2 \) (this is generally a more accurate way to model air resistance). Solve the resulting ODE for the velocity of the object.

6. With values of \( m = 100 \) kg and \( k = .1 \) kg/meter, what is the terminal velocity of the object? (Notice that the \( k \)'s that appear in the two models are different.)

7. Find the function that describes the position \( x(t) \) of the object for all assuming that \( x = 0 \) corresponds to the position at which the object is dropped.

8. Make rough sketches of \( v(t) \) and \( x(t) \).

9. In at least two paragraphs, compare and contrast the two models for air resistance.

**Project 3 - Periodic Drug Doses**

Most drugs are eliminated from the body according to a strict exponential decay law. Here are two problems that illustrate the process.
1. The drug Valium has a half-life in the blood (a population average) of 36 hours. Assume that a 50-milligram dose of Valium is taken at time $t = 0$. Let $m(t)$ be the amount of drug in the blood in milligrams $t$ hours after the dose. Plot the function $m(t)$ as it varies with time. After how many hours, will the amount of drug reach 10% of its initial value? After how many hours, will the amount of drug reach 1% of its initial value?

2. Now imagine that a drug (such as aspirin or an antibiotic) with a half-life of 12 hours is taken regularly every eight hours. Assume that the first dose is taken at time $t = 0$. Draw A rough sketch of the amount of drug in the blood at time $t$.

(a) What is the amount of drug in the blood at $t = 8$ hours just prior to the second dose?
(b) What is the amount of drug in the blood at $t = 8$ hours just after to the second dose?
(c) What is the amount of drug in the blood at $t = 16$ hours just prior to the third dose?
(d) What is the amount of drug in the blood at $t = 16$ hours just after to the third dose?
(e) Now generalize: what is the amount of drug in the blood at $t = 8(n - 1)$ hours just prior to the $n$th dose, where $n = 1, 2, 3, \ldots$?
(f) What is the amount of drug in the blood at $t = 8(n - 1)$ hours just after to the $n$th dose?
(g) What can you say about the long-term amount of the drug in the blood? Does it continue to increase without bound or does it approach a steady-state level? If you argue for the latter choice, find the steady-state value of the amount of drug. Justify your conclusion carefully.

(h) Quickly apply the periodic doses problem to the following problem: A fish hatchery harvests $\frac{1}{3}$ of its current fish population at the end of each year, and then immediately replenishes the population with 500 new fish. Assuming no deaths and an initial fish population of 1000 fish, what is the steady state population in the hatchery?

Project 4 - Cannon Design
The basic design of a cannon with a stationary carriage and a sliding gun tube-breech block assembly (hereafter called gun assembly) is shown in the figure below. A damping piston between the carriage and the gun assembly absorbs the recoil of the firing. A recoil spring, in a similar position, is designed to push the gun assembly back into the firing position. Assume that the mass of the gun assembly is $m$ kilograms, the recoil spring has a spring constant of $k$ newtons/meter, and the force exerted by the damping mechanism (in newtons) is $B$ times the velocity of the gun assembly. Assume also that when the cannon is fired, an instantaneous velocity of is imparted to the gun tube.
1. Model the horizontal displacement of the gun assembly with a second-order ODE with initial conditions.

2. Find the solution of the initial value problem with the parameter values $m = 1500$ kg, $k = 19,500$ newton/meter, $B = 9000$ newton-sec/meter, and $v_0 = 5$ meter/sec.

3. Graph the motion of the gun assembly for the first three seconds after firing.

4. At what time does the gun assembly first return to the firing position? What is the velocity of the assembly at that time?

5. At what time does the gun assembly reach its maximum displacement? What is that displacement?

Project 5 - Mixing Tank Reactions

1. A large tank is filled with 500 liters of pure water. At time $t = 0$, an inflow valve is opened and a brine solution with a concentration of 500 grams of salt per liter flows into the tank at 5 liters/minute. At the same time ($t = 0$), an outflow valve is opened and the thoroughly mixed solution in the tank flows out of the tank at 5 liters/minute.

   (a) Derive the ODE that describes either the concentration of the solution in the tank or the mass of salt in the tank for all times $t \geq 0$.

   (b) Solve this ODE together with the appropriate initial condition.

   (c) Graph and interpret the solution. In particular, what can you say about the concentration in the tank as $t \to \infty$?

2. Now imagine that the experiment of part (1) is repeated, except that at $t = 0$, the mixing tank springs a leak and the thoroughly mixed solution also flows out through the leak at a rate of .5 liter/minute.

   (a) Derive the ODE that describes either the mass of salt in the tank for all times $t \geq 0$.

   (b) Solve this ODE together with the appropriate initial condition.

   (c) Find the function that gives the concentration of salt in the tank for all times $t \geq 0$.

   (d) Graph and interpret both the mass and concentration functions.

   (e) At what time does the mass function have a maximum?

   (f) When does the tank become empty and what is the concentration of the solution in the tank just as the tank becomes empty?
Project 6 - Period of the Pendulum

The full equation of motion for the undamped pendulum is

\[ \theta''(t) + \omega^2 \sin \theta(t) = 0, \]

where \( \omega^2 = g/l \), \( g \) is the acceleration of gravity, \( l \) is the length of the pendulum, \( \theta \) and is measured in radians. (The more familiar linear equation results by assuming that \( \theta << 1 \) (small amplitudes) and that \( \sin \theta \approx \theta \).) An explicit solution of this nonlinear ODE cannot be found in terms of familiar functions. However, it is possible to determine the period of the nonlinear pendulum. Assume the initial conditions \( \theta(0) = \alpha \) and \( \theta'(0) = 0 \). We will consider the quarter period when \( \theta \) decreases from \( \theta = \alpha \) to \( \theta = 0 \).

1. What is the period of the linear pendulum (governed by \( \theta'' + \omega^2 \theta = 0 \) )?
2. Now let’s work on the full nonlinear ODE. Multiply both sides of the ODE by \( \theta' \), use the chain rule carefully, and apply the initial conditions to show that

\[ \theta'(t) = \pm \omega \sqrt{2(\cos \theta(t) - \cos \alpha)} \]

(Note that \( (\theta')^2 = \theta'' \).)
3. Do you choose the plus or minus branch of the square root? During the quarter period we are considering, is \( \theta' < 0 \) or is \( \theta' > 0 \)?
4. Now separate variables and write

\[ dt = -\frac{1}{\omega} \frac{d\theta}{\sqrt{2(\cos \theta - \cos \alpha)}}. \]

5. The identity \( \cos x = 1 - 2 \sin^2 x/2 \) should lead you to

\[ dt = -\frac{1}{2\omega} \frac{d\theta}{\sqrt{\sin^2 (\alpha/2) - \sin^2 (\theta/2)}}. \]

6. The function on the right is still difficult to integrate, so we define a new variable \( \phi \) by \( \sin \theta/2 = \sin (\alpha/2) \sin \phi \). After changing variables you should have

\[ dt = -\frac{1}{\omega} \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}}, \]

where \( k = \sin (\alpha/2) \).
7. Now we can integrate. Notice that when the original variable \( \theta \) varies from \( \theta = \alpha \) to \( \theta = 0 \), the new variable \( \phi \) varies from \( \phi = \pi/2 \) to \( \phi = 0 \). Letting \( T \) be the full period and integrating over a quarter period, show that

\[ T/4 = \frac{1}{\omega} \int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}}. \]
8. This last integral is called an elliptic integral of the first kind and is denoted as $F(k, \pi/2)$. Therefore, we have found that the period of the pendulum is $T = (4/\pi)F(k, \pi/2)$.

9. What is the period in the limiting case $k=0$ which corresponds to $\alpha = 0$.

10. Use a table of elliptic integrals to find the period of a pendulum with $l = 2$ meters with initial displacements of $\alpha = \pi/12, \pi/6, \pi/3, \pi/2$.

**Project 7 - Population Genetics**

You may recall from a biology course that the simplest genetic traits are transmitted by a single gene that has one of two forms or alleles. For example, a gene for eye color may be $A$ for the dominant allele (brown eyes) or $a$ for the recessive allele (blue eyes). With one gene coming from each parent, an offspring may have one of three genotypes: $AA$ (brown eyes), $Aa$ (brown eyes), or $aa$ (blue eyes). Suppose that mating takes place randomly (the genes are thoroughly mixed) and that the three genotypes have different fitnesses (or survival probabilities), $b$, $c$, and $d$, respectively. We will let $p(t)$ represent the fraction of $A$ alleles in the gene pool at time $t$. This means that $0 \leq p \leq 1$ and the fraction of $a$ alleles is $1 - p$. The differential equation that governs how $p$ varies in time is

$$p'(t) = p(1-p)((b - 2c + d)p + c - d). \quad (9)$$

1. What are the equilibrium points for this ODE?

2. Draw the direction field for the case that $b = 0.2$, $c = 0.4$ and $d = 0.6$ and determine the stability of the equilibrium points. Determine $\lim_{t \to \infty} p(t)$.

3. Draw the direction field for the case that $d = 0.2$, $c = 0.4$ and $b = 0.6$ and determine the stability of the equilibrium points. Determine $\lim_{t \to \infty} p(t)$.

4. Draw the direction field for the case that $b = 0.2$, $d = 0.4$ and $c = 0.6$ and determine the stability of the equilibrium points. Determine $\lim_{t \to \infty} p(t)$.

5. Draw the direction field for the case that $c = 0.2$, $b = 0.4$ and $d = 0.6$ and determine the stability of the equilibrium points. Determine $\lim_{t \to \infty} p(t)$.

6. Find the general conditions on $b$, $c$, and $d$ for polymorphism, the coexistence of both genes in the steady state.

**Project 8 - A Pursuit Problem** A dog walks north from a crossroads at 1 mile per hour. The dog’s master begins one mile east of the crossroads and walks at all times directly at the dog with a speed of $s > 1$ miles per hour.
1. Find the equation (in the form \( y = f(x) \)) that describes the path of the master.

2. When and where does the master overtake the dog if \( s = 1.5 \)?

3. Find the function that gives the meeting time in terms of \( s \).

Project 9 - Coding and Information Content

Imagine a coding system in which messages can be formed from a short signal (S) with duration of one time unit and a long signal (L) with a duration of two time units. For example, a message of the form SSLSSS, would have a duration of six time units. Let \( M_n \) be the number of different ways of forming a message of length \( n \). For example, \( M_4 = 5 \) since the messages SSL, LL, SSSS, SLS, and LSS all have a duration of four time units.

1. What are \( M_1 \) and \( M_2 \)?

2. Show carefully and convincingly that \( M_{n+1} = M_n + M_{n-1} \).

3. Assume a trial solution of the form \( M_n = \lambda^n \), where \( \lambda \) is constant to be determined. Proceed just as you would with a second order constant coefficient ODE and solve this difference equation for \( M_n \), where \( n = 1, 2, 3, \ldots \).

4. Without computing 100 terms of the sequence \( M_n \), what is the value of \( M_{100} \)?

5. The capacity of a channel is defined (by Claude Shannon, the innovator of information theory) to be

\[
C = \lim_{n \to \infty} \frac{\log_2 M_n}{n}
\]  

(10)

Find the capacity of the channel carrying messages of the form described above.

Project 10 - A Predator-Prey Model

The Lotka-Volterra model for describing the interaction between a predator and a prey was formulated in the early part of this century. It has been shown to be fairly accurate when applied to many natural systems (lynx-rabbits, sharks-fish). Let \( F(t) \) and \( R(t) \) denote the population of a predator and a prey species (think Fox and Rabbit) at time \( t \), measured in hundred of individuals. Consider the predator-prey equations

\[
F'(t) = -6F + 2FR
\]

(11)

\[
R'(t) = 12R - 3FR.
\]

(12)
1. First give a brief interpretation of the equations. What is the effect of an increase in the rabbit population on the existing rabbit and fox populations? What is the effect of an increase in the fox population on the existing rabbit and fox populations?

2. For what fox and rabbit populations is the system at equilibrium?

3. Sketch the direction field for these equations in the phase plane. (You need to consider only $R > 0$ and $F > 0$. Why?) Choose a few different initial conditions and sketch the resulting trajectories in the phase plane.

4. Divide the predator ODE by the prey ODE (or vice-versa) to obtain a single ODE in $R$ and $F$ (with $t$ no longer present). Solve this separable ODE to find an implicit representation for the trajectories.

5. Give the most convincing argument possible that this system has periodic solutions; that is, there is a time interval of length $T$ such that $R(t+T) = R(t)$ and $F(t+T) = F(t)$.

6. Show that that the average populations are given by $\bar{R} = 3$ and $\bar{F} = 4$.

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**Project 11 - Kepler’s Laws of Planetary Motion**

Newton’s law of gravitation states that two mass points, $m_1$ and $m_2$, attract each other with a force proportional to the product of the masses and inversely proportional to the square of the distance between them. That is,

$$F = \frac{Gm_1m_2}{r^2},$$

where $G$ is the gravitational constant and $r$ is the distance between the two masses.

Johannes Kepler (1571-1630), Kepler’s Laws:

1. Orbits are ellipses with the sun at one focus.

2. The radius of the Sun to the planet sweeps out equal areas in equal times.

3. The square of the time for one traverse of an orbit is proportional to the cube of the mean distance to the sun.

We make several simplifications and assumptions to make the problem tractable.

1. Assume that the planets are perfect spheres with mass concentrated at the center, i.e. assume that planets are point-masses.
2. Assume that the Earth and the Sun are in perfect isolation. Neglect the presence of all other planets.

3. Assume that the mass of the planet Earth is negligible in relation to the mass of the Sun. In other words, the Sun is not moved by the Earth.

Given the assumptions and simplifications above, it can be shown that the orbit of the planet lies in a plane. See Figure 1. Newton’s Second Law states that force equals mass time acceleration, $F = m \cdot a$, which can be decomposed into the $x$ and $y$ directions as follows.

<table>
<thead>
<tr>
<th>In the $x$-direction</th>
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<td>$mx'' = mf_x$</td>
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<td>$x'' = f_x$</td>
<td>$y'' = f_y$</td>
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Let $w = x + iy$ denote the position of the Earth, where $i = \sqrt{-1}$. Thus,

$$w'' = x'' + iy'' = f_x + if_y = F. \quad (13)$$

The natural setting for this problem is polar coordinates $(r, \theta)$, where $x = r \cos \theta$ and $y = r \sin \theta$.

1. Use the Maclaurin series for $\cos \theta$ and $\sin \theta$ to prove Euler’s relation

$$e^{i\theta} = \cos \theta + i \sin \theta.$$ 

2. Write $w$ in polar coordinates and use Euler’s relation to obtain

$$w = re^{i\theta}. \quad (14)$$ 

3. Since $F$ is directed toward the Sun at all times,

$$F = Re^{i\theta}, \quad (15)$$

where $R = R(r, \theta)$ is the magnitude of the force. A central field is one in which the force is directed toward a fixed point. Use equations (13), (14), and (15) to derive the equation for plane motion in a central force field

$$(re^{i\theta})'' = Re^{i\theta}. \quad (16)$$

Next, expand the left-hand side of (16) and equate the real and imaginary parts to obtain

$$r'' - r(\theta')^2 = R \quad (17)$$

$$(r\theta')' + r\theta' = 0. \quad (18)$$
4. Multiply equation (18) through by $r$ to obtain

$$r^2 \dot{\theta} = C,$$

where $C$ is a constant. Note that if $C = 0$ then the orbit is a straight line, a case that we exclude for now. Note that this equation is a first-order, linear differential equation which happens to be separable. Multiply through by $\frac{1}{2}$ and integrate to obtain Kepler’s Second Law

$$\int_{\theta_1}^{\theta_2} \frac{1}{2} r^2 \, d\theta = \int_{t_1}^{t_2} \frac{C}{2} \, dt. \quad (19)$$

In a short paragraph (3-5 sentences), give a physical interpretation of this equation.

5. Kepler’s second law is valid for motion in any central force field. Recall Newton’s law of gravitation from the beginning. Under the given assumptions this relation can be simplified to say that the magnitude of the force is inversely proportional to $r^2$, that is, $R = -\frac{k}{r^2}$, where $k > 0$ and the minus sign signifies that the force is directed toward the origin (the Sun). Substitute this into equation (17) to obtain the following second-order, non-linear, non-homogeneous differential equation

$$r'' - \frac{C^2}{r^3} = -\frac{k}{r^2}. \quad (20)$$

6. Perform a transformation of variables $s = \frac{1}{r}$ on equation (20) to obtain a the second-order, linear, non-homogeneous differential equation with constant coefficients

$$\frac{d^2s}{d\theta^2} + s = \alpha, \quad (21)$$

where $\alpha = \frac{k}{r^2}$. The solution to this equation can be written

$$s = \alpha - \beta \cos(\theta - \theta_0), \quad (22)$$

where $\beta$ and $\theta_0$ are constants.

7. Transform equation (22) back from $s$ to $r$ to get

$$r = \frac{1}{\alpha - \beta \cos(\theta - \theta_0)}.$$ 

Perhaps the easiest way to see that this is the equation of a conic section with one focus at the origin is to...

8. Kepler’s third law depends on the assumption that the mass of the sun is much larger than the mass of the planet.

(a) If the mass of the Sun is $1.98892 \times 10^{30}$ and the mass of the Earth is $5.9742 \times 10^{24}$, how many times more massive is the Sun than the Earth.

(b) What does this assumption mean with respect to the constant of proportionality, $k$, in $R = -\frac{k}{r^2}$?