Find the series solutions in the closed form.

$$x^2y'' + 2xy' + xy = 0$$

Solution We have the ode

$$x^2y'' + 2xy' + xy = 0 \quad ---- (1)$$

Note that $P(x) = x^2 = 0$ when $x=0$. So $x=0$ is a singular point. Since

$$\lim_{x\to 0} x \cdot \frac{2x}{x^2} = 2, \text{ and}$$

$$\lim_{x\to 0} x^2 \cdot \frac{x}{x^2} = 0, \text{ we get that}$$

$x=0$ is a regular singular point of (1).

Let $y = \sum_{n=0}^{\infty} a_n x^{n+r}$ -----(2) be a solution.

Then $y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$ --- (3),

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} \quad --- (4).$$

Substituting (2), (3), (4) into (1), we get

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r} + \sum_{n=0}^{\infty} 2(n+r)a_n x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r+1} = 0$$

$$r(r-1)a_0x^r + 2ra_0x^r + \sum_{n=1}^{\infty}\left[\{(n+r)(n+r-1)+2(n+r)\}a_n + a_{n-1}\right]x^{n+r} = 0$$

$$(r+1)ra_0x^r + \sum_{n=1}^{\infty}\left[(n+r+1)(n+r)a_n + a_{n-1}\right]x^{n+r} = 0$$

$(a_0=1)$

$$\Rightarrow (r+1)r = 0, \quad a_n = -\frac{a_{n-1}}{(n+r+1)(n+r)}, \quad n \geq 1$$

Let $F(r) := r(r+1)$. Then we have (from the last page)

$$\begin{cases} F(r) = 0 \Rightarrow r = 0, -1. \\ a_n = -\dfrac{a_{n-1}}{F(r+n)}, \quad n \geq 1. \leftarrow \text{Recurrence relation} \end{cases}$$

From the Recurrence Relation,

$$a_1 = -\frac{a_0}{F(r+1)} = -\frac{1}{F(r+1)} \quad (\text{taking } a_0 = 1)$$

$$a_2 = -\frac{a_1}{F(r+2)} = \frac{1}{F(r+1)\,F(r+2)}$$

$$a_3 = \frac{1}{F(r+1)\,F(r+2)\,F(r+3)}$$

$$\vdots$$

$$a_n = (-1)^n \frac{1}{F(r+1)\,F(r+2)\cdots F(r+n)}, \quad n \geq 1 \text{ --- } (5)$$

For $r = r_1 = 0$ (bigger root), eqn (5) becomes

$$a_n = (-1)^n \frac{1}{F(1)\,F(2)\cdots F(n)}, \quad n \geq 1$$

$$= (-1)^n \frac{1}{1\cdot 2\cdot 2\cdot 3\cdot 3\cdot 4\cdots n(n+1)}, \quad n \geq 1$$

$$= (-1)^n \frac{1}{n!\,(n+1)!}, \quad n \geq 0 \quad (a_0 = 1)$$

Thus a solution ($r = r_1 = 0$) is

$$y = y_1(x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{n!\,(n+1)!} x^n \text{ ---- } (6)$$

Since $r_1 - r_2 = 0-(-1) = 1$ $(=N)$, the second solution has the form

$$y = y_2(x) = a\, y_1(x) \ln|x| + |x|^{r_2}\left[1 + \sum_{n=1}^{\infty} c_n(r_2)\, x^n\right] \quad \text{---}(7)$$

where $a = \lim_{r\to r_2} (r-r_2)\, a_N(r)$ $\;\cdots(8)$, and

$$c_n(r_2) = \frac{d}{dr}\left[(r-r_2)\, a_n(r)\right]\Big|_{r=r_2}, \; n \geq 1 \quad \cdots (9).$$

Now from (8) with $r_2 = -1$, $N=1$,

$$a = \lim_{r\to -1} (r+1)\, a_1(r)$$

$$= \lim_{r\to -1} (r+1) \cdot -\frac{1}{F(r+1)} \quad \text{(using (5))}$$

$$= \lim_{r\to -1} (r+1) \cdot -\frac{1}{(r+1)(r+2)} = -1,$$

and

$$c_n(-1) = \frac{d}{dr}\left[(r+1) \cdot (-1)^n \frac{1}{F(r+1)F(r+2)\cdots F(r+n)}\right]\Bigg|_{r=-1}$$

$$= (-1)^n \frac{d}{dr}\left[\frac{\cancel{(r+1)}}{\cancel{(r+1)}(r+2)\cdot(r+2)(r+3)\cdot \cdots\, (r+n)(r+n+1)}\right]\Bigg|_{r=-1}$$

$$= (-1)^n \frac{d}{dr}\left[\frac{1}{(r+2)^2(r+3)^2\cdots(r+n)^2(r+n+1)}\right]\Bigg|_{r=-1} \quad \text{---}(10)$$

We use the following fact to evaluate (10):

If $f(x) = (x-a_1)^{m_1}(x-a_2)^{m_2} \cdots (x-a_n)^{m_n}$,

$$\frac{f'(x)}{f(x)} = \frac{m_1}{x-a_1} + \frac{m_2}{x-a_2} + \cdots + \frac{m_n}{x-a_n}.$$

Assume $G_n(r) = (r+2)^2(r+3)^2 \cdots (r+n)^2(r+n+1)$,

then $$\frac{G_n'(r)}{G_n(r)} = \frac{2}{r+2} + \frac{2}{r+3} + \cdots + \frac{2}{r+n} + \frac{1}{r+n+1}.$$

Now from equation (10), we have

$$a_n = (-1)^n \frac{d}{dr}\left[G_n(r)\right]\Big|_{r=-1}$$

$$= (-1)^n \cdot -\frac{G_n'(r)}{G_n^2(r)}\Big|_{r=-1}$$

$$= (-1)^{n+1} \frac{G_n'(r)}{G_n(r)} \cdot \frac{1}{G_n(r)}\Big|_{r=-1}$$

$$= (-1)^{n+1}\left[\frac{2}{r+2} + \frac{2}{r+3} + \cdots + \frac{2}{r+n} + \frac{1}{r+n+1}\right] \cdot \frac{1}{(r+2)^2(r+3)^2 \cdots (r+n+1)}\Big|_{r=-1}$$

$$= (-1)^{n+1}\left[\frac{2}{1} + \frac{2}{2} + \frac{2}{3} + \cdots + \frac{2}{n-1} + \frac{1}{n}\right] \cdot \frac{1}{1^2 \cdot 2^2 \cdots (n-1)^2 \cdot n}$$

$$= (-1)^{n+1}\left[\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1}\right) + \left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right)\right] \cdot \frac{1}{(n-1)!\, n!}$$

$$= (-1)^{n+1}\left[H_{n-1} + H_n\right] \cdot \frac{1}{(n-1)!\, n!},$$ where $H_m := 1 + \frac{1}{2} + \cdots + \frac{1}{m}$.

Thus the second solution is

$$y = y_2(x) = -y_1(x) \ln|x| + \frac{1}{|x|}\left[1 + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{H_{n-1} + H_n}{(n-1)!\, n!} x^n\right].$$