Solve: $x(x-1)y'' + 6x^2y' + 3y = 0$ about $x_0 = 0$.

Note that $x = 0$ is a singular pt.

Also $p_0 = \lim_{x\to 0} x\,p(x) = \lim_{x\to 0} x \dfrac{6x^2}{x(x-1)} = 0$

$q_0 = \lim_{x\to 0} x^2 q(x) = \lim_{x\to 0} x^2 \dfrac{3}{x(x-1)} = 0,$

Thus $x=0$ is a regular singular point.

Indicial equation: $r(r-1) + p_0 r + q_0 = 0$

i.e., $r(r-1) = 0$

$\Rightarrow r_1 = 1,\ r_2 = 0.$

First consider $r = r_1 = 1$.

Let $y = \phi(x,r) = x^r \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+1}$ be a soln.

Then $y' = \sum_{n=0}^{\infty} (n+1) a_n x^n$

$y'' = \sum_{n=1}^{\infty} n(n+1) a_n x^{n-1}$.

With these the ode becomes

$$\sum_{n=1}^{\infty} n(n+1) a_n x^{n+1} - \sum_{n=1}^{\infty} n(n+1) a_n x^n + 6\sum_{n=0}^{\infty} (n+1) a_n x^{n+2} + 3\sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

or, $$\sum_{n=2}^{\infty} (n-1)n\, a_{n-1} x^n - \sum_{n=1}^{\infty} n(n+1) a_n x^n + 6\sum_{n=2}^{\infty} (n-1) a_{n-2} x^n + 3\sum_{n=1}^{\infty} a_{n-1} x^n = 0$$

or, $$\sum_{n=2}^{\infty} (n-1)n\, a_{n-1} x^n - (2a_1 x) - \sum_{n=2}^{\infty} n(n+1) a_n x^n + 6\sum_{n=2}^{\infty} (n-1) a_{n-2} x^n + (3a_0 x) + 3\sum_{n=2}^{\infty} a_{n-1} x^n = 0$$

$\rightarrow$

or, $(-2a_1+3a_0)x+\sum_{n=2}^{\infty}\left[((n-1)n+3)a_{n-1}-n(n+1)a_n+6(n-1)a_{n-2}\right]x^n=0$

$$\Rightarrow \begin{cases} -2a_1+3a_0=0 \quad \text{or} \quad a_1=\frac{3}{2}a_0. \\ (n^2-n+3)a_{n-1}-n(n+1)a_n+6(n-1)a_{n-2}=0, \quad n\geq 2 \end{cases}$$

$$\text{or} \quad a_n=\frac{(n^2-n+3)a_{n-1}+6(n-1)a_{n-2}}{n(n+1)}, \quad n\geq 2.$$

(Recurrence relation)

Take $a_0=1$, we get

$a_1=\frac{3}{2}\cdot 1=\frac{3}{2}$

(n=2) $a_2=\frac{5a_1+6a_0}{2\cdot 3}=\frac{5\cdot\frac{3}{2}+6\cdot 1}{6}=\frac{9}{4}$

(n=3) $a_3=\frac{51}{16}$

(n=4) $a_4=\frac{111}{40}$

Thus one solution is

$$y=y_1(x)=\sum_{n=0}^{\infty}a_n x^{n+1}$$
$$=a_0x+a_1x^2+a_2x^3+a_3x^4+a_4x^5+\cdots$$
$$=x+\frac{3}{2}x^2+\frac{9}{4}x^3+\frac{51}{16}x^4+\frac{111}{40}x^5+\cdots$$

Since $r_1-r_2=1$, a positive integer, the second solution has the form

$$y=y_2(x)=a\,y_1(x)\ln x+x^{r_2}\left[1+\sum_{n=1}^{\infty}c_n(r_2)x^n\right]$$

$(r_2=0)$ i.e., $y=a y_1\ln x+\left[1+\sum_{n=1}^{\infty}c_n x^n\right]$

Then $y'=a y_1'\ln x+a y_1\cdot\frac{1}{x}+\left[1+\sum_1^{\infty}c_n x^n\right]'$

$y''=a y_1''\ln x+2a y_1'\frac{1}{x}-a y_1\cdot\frac{1}{x^2}+\left[1+\sum_1^{\infty}c_n x^n\right]''$

→

Substituting y, y', y'' into the ode, we get

$$x(x-1)\left[a y_1'' \ln x + 2a y_1' \frac{1}{x} - a y_1 \frac{1}{x^2}\right] + x(x-1)\left[1+\sum_1^\infty c_n x^n\right]''$$
$$+ 6x^2\left[a y_1' \ln x + a y_1 \frac{1}{x}\right] + 6x^2\left[1+\sum_1^\infty c_n x^n\right]'$$
$$+ 3a y_1 \ln x + 3\left[1+\sum_1^\infty c_n x^n\right] = 0$$

0 as y_1 is a soln.

or, $$a \ln x\left[x(x-1)y_1'' + 6x^2 y_1' + 3y_1\right] + 2a(x-1)y_1' - a y_1 \frac{(x-1)}{x}$$
$$+ 6x a y_1 + L\left[1+\sum_1^\infty c_n x^n\right] = 0$$

$$\left(L[y] := x(x-1)y'' + 6x^2 y' + 3y\right)$$

$$\Rightarrow L\left[1+\sum_1^\infty c_n x^n\right] = -2axy_1' + 2ay_1' + ay_1 - ay_1\frac{1}{x} - 6axy_1 \quad (*)$$

Let's consider the left side only:

$$L\left[1+\sum_1^\infty c_n x^n\right] = x(x-1)\left[1+\sum_1^\infty c_n x^n\right]'' + 6x^2\left[1+\sum_1^\infty c_n x^n\right]' + 3\left[1+\sum_1^\infty c_n x^n\right]$$

$$= x(x-1)\left[1+c_1x+c_2x^2+c_3x^3+\ldots\right]''$$
$$+ 6x^2\left[1+c_1x+c_2x^2+c_3x^3+\cdots\right]'$$
$$+ 3\left[1+c_1x+c_2x^2+c_3x^3+\cdots\right]$$

$$= x(x-1)\left[2c_2 + 6c_3x + \cdots\right] + 6x^2\left[c_1+2c_2x+3c_3x^2+\cdots\right]$$
$$+ 3\left[1+c_1x+c_2x^2+c_3x^3+\cdots\right]$$

$$= 2(x^2-x)c_2 + 6(x^3-x^2)c_3 + \cdots + 6c_1x^2 + 12c_2x^3 + \cdots$$
$$+ 3 + 3c_1x + 3c_2x^2 + 3c_3x^3 + \cdots$$

$$= 3 + (3c_1 - 2c_2)x + (2c_2 + 6c_1 + 3c_2 - 6c_3)x^2 + \cdots$$

$$= 3 + (3c_1 - 2c_2)x + (6c_1 + 5c_2 - 6c_3)x^2 + \cdots$$

Now we consider the right side of (*), we have

$$-2axy_1' + 2ay_1' + ay_1 - ay_1\frac{1}{x} - 6axy_1$$

$$= -2ax\left(x+\frac{3}{2}x^2+\cdots\right)' + 2a\left(x+\frac{3}{2}x^2+\frac{9}{4}x^3+\cdots\right)'$$
$$+a\left(x+\frac{3}{2}x^2+\cdots\right) - \frac{a}{x}\left(x+\frac{3}{2}x^2+\frac{9}{4}x^3+\cdots\right)$$
$$-6ax\left(x+\frac{3}{2}x^2+\cdots\right)$$

$$= \cdots\cdots$$

$$= a + \frac{7}{2}ax + \frac{3}{4}ax^2 + \cdots\cdots$$

Now from (*), we get (Equating both sides)

$$\left.\begin{aligned} 3 &= a \\ 3c_1 - 2c_2 &= \frac{7}{2}a \\ 6c_1 + 5c_2 - 6c_3 &= \frac{3}{4}a \end{aligned}\right\} \text{solve (set } c_1 = 0)$$

$\vdots$

$a = 3$

$c_1 = 0$ (otherwise we can't solve)

$c_2 = -\frac{21}{4}$

$c_3 = -\frac{19}{4}$

$\vdots$

Hence a second solution is

$$y = y_2(x) = 3y_1(x)\ln x + \left[1 - \frac{21}{4}x^2 - \frac{19}{4}x^3 - \cdots\right].$$