Example: Find the solution of the heat conduction problem

$$u_t = u_{xx} + x e^{-t}$$
$$u(0,t) = 0, \quad u_x(1,t) + u(1,t) = 0$$
$$u(x,0) = 0$$

$r(x) = 1$
$f(x) = 0.$
$F(x,t) = x e^{-t}$

Solution:

$$u(x,t) = \sum_{n=1}^{\infty} b_n(t)\, \phi_n(x)$$

where ϕ_n's are the normalized eigenfunctions of

$$X'' + \lambda X = 0, \quad X(0) = 0, \quad X'(1) + X(1) = 0 \text{ --- } (1)$$

(Find the eigenvalues/eigenfunctions of (1) and normalize the eigenfunctions)

$$\phi_n(x) = \frac{\sqrt{2}\sin\sqrt{\lambda_n}\, x}{\sqrt{1+\cos^2\sqrt{\lambda_n}}},$$

λ_n satisfies
$\sin\sqrt{\lambda_n} + \sqrt{\lambda_n}\cos\sqrt{\lambda_n} = 0$
and $\lambda_n \approx \frac{(2n-1)^2\pi^2}{4}$
for large n.

To find $b_n(t)$, we first find

$$B_n = \int_0^1 r(x) f(x)\, \phi_n(x)\, dx = 0, \quad (\text{as } f(x) = 0)$$

$$\gamma_n(t) = \int_0^1 F(x,t)\, \phi_n(x)\, dx$$

$$= \int_0^1 x e^{-t} \frac{\sqrt{2}\sin\sqrt{\lambda_n}\, x}{\sqrt{1+\cos^2\sqrt{\lambda_n}}}\, dx$$

$\longrightarrow$

$$= \frac{\sqrt{2}e^{-t}}{\sqrt{1+\cos^2\sqrt{\lambda_n}}} \underbrace{\int_0^1 x \sin\sqrt{\lambda_n}\, x \, dx}_{\text{IBP}}$$

$u = x \quad dv = \sin\sqrt{\lambda_n}\, x$

$du = 1\,dx \quad v = -\frac{1}{\sqrt{\lambda_n}} \cos\sqrt{\lambda_n}\, x$

.....

$$= \frac{\sqrt{2}e^{-t}}{\sqrt{1+\cos^2\sqrt{\lambda_n}}} \left[-\frac{1}{\sqrt{\lambda_n}} x \cos\sqrt{\lambda_n}\, x \Big|_0^1 + \frac{1}{\sqrt{\lambda_n}} \frac{\sin\sqrt{\lambda_n}\, x}{\sqrt{\lambda_n}} \Big|_0^1 \right]$$

$$= \frac{\sqrt{2}\, e^{-t}}{\sqrt{1+\cos^2\sqrt{\lambda_n}}} \left[-\underbrace{\frac{\cos\sqrt{\lambda_n}}{\sqrt{\lambda_n}}}_{= \frac{\sin\sqrt{\lambda_n}}{\lambda_n}} + \frac{\sin\sqrt{\lambda_n}}{\lambda_n} \right]$$

(since $\sin\sqrt{\lambda_n} + \sqrt{\lambda_n}\cos\sqrt{\lambda_n} = 0$)

$$= \frac{2\sqrt{2}e^{-t}\sin\sqrt{\lambda_n}}{\lambda_n\sqrt{1+\cos^2\sqrt{\lambda_n}}} = K^* e^{-t}, \text{ where } K^* = \frac{2\sqrt{2}\sin\sqrt{\lambda_n}}{\lambda_n\sqrt{1+\cos^2\sqrt{\lambda_n}}}$$

Now $b_n = B_n e^{-\lambda_n t} + \int_0^t e^{-\lambda_n(t-s)} \nu_n(s)\, ds, \quad n \geq 1$

$$= 0 + \int_0^t e^{-\lambda_n t} \cdot e^{\lambda_n s} \cdot K^* e^{-s}\, ds$$

$$= \cdots\cdots = K^* \frac{e^{-t} - e^{-\lambda_n t}}{\lambda_n - 1}.$$

Thus $u(x,t) = \sum_{n=1}^{\infty} b_n(t)\, \phi_n(x) = \cdots\cdots \leftarrow \boxed{\text{Answer}}$