Solve the following nonhomogeneous heat equation problem

$$u_t = u_{xx} + xe^{-t}, \quad u(0,t)=0, \; u(1,t)=0$$
$$u(x,0) = 5.$$

Soln: Comparing this system with the general system, we get

$$r(x)=1, \quad f(x)=5, \quad F(x,t) = xe^{-t}$$

We want to find the eigenvalues/eigenfunctions of the corresponding eigenvalue problem

$$X'' + \lambda X = 0, \quad X(0)=0, \; X(1)=0.$$

We have found them as

$$X_n(x) = \sin(n\pi x), \quad \lambda_n = n^2\pi^2 \quad \text{for } n \geq 1$$

Solution is $u(x,t) = \sum_{n=1}^{\infty} b_n(t)\,\phi_n(x)$. For we normalize the eigenfunctions $\{\sin(n\pi x)\}$, we get $\phi_n(x) = \sqrt{2}\sin(n\pi x)$, $n \geq 1$.

Next we find B_n as

$$B_n = \int_0^1 r(x)\,f(x)\,\phi_n(x)\,dx = \frac{5\sqrt{2}[1-(-1)^n]}{n\pi}.$$

$\longrightarrow$

Next $\nu_n(t) = \int_0^1 x e^{-t} \sqrt{2} \sin(n\pi x)\, dx$

$$= \sqrt{2} e^{-t} \int_0^1 x \sin(n\pi x)\, dx$$

$$= \sqrt{2} e^{-t} \frac{(-1)^{n+1}}{n\pi} \quad \text{(IBP)}$$

$$= K_n e^{-t}, \quad \text{where } K_n = \frac{\sqrt{2}(-1)^{n+1}}{n\pi} \text{ (say)}$$

So $b_n(t) = B_n e^{-\lambda_n t} + \int_0^t e^{-\lambda_n (t-s)} \nu_n(s)\, ds$

$$= B_n e^{-\lambda_n t} + K_n e^{-\lambda_n t} \int_0^t e^{\lambda_n s} e^{-s}\, ds$$

$$= B_n e^{-\lambda_n t} + K_n e^{-\lambda_n t} \int_0^t e^{(\lambda_n - 1)s}\, ds$$

$$= B_n e^{-\lambda_n t} + K_n e^{-\lambda_n t} \left(e^{(\lambda_n - 1)t} - 1\right) / (\lambda_n - 1)$$

$$= B_n e^{-\lambda_n t} + K_n \left(e^{-t} - e^{-\lambda_n t}\right) / (\lambda_n - 1)$$

Hence the solution is

$$u(x,t) = \sum_{n=1}^{\infty} b_n(t)\, \phi_n(x) \quad \text{where}$$

$\phi_n(x)$ and $b_n(t)$ are found above.