Here the ODE is $-y'' = 2y + x$.

Consider $y'' + \lambda y = 0$ with the boundary conditions $y(0) = 0$, $y'(1) = 0$. $(\mu = 2)$

Case I: $\lambda < 0$. Let $\lambda = -\mu^2$ $(\mu > 0)$. Then the ODE

$$y'' - \mu^2 y = 0.$$

char. eqn. $r^2 - \mu^2 = 0 \Rightarrow r = \pm\mu$

solution: $y(x) = c_1 \cosh(\mu x) + c_2 \sinh(\mu x)$

and $y'(x) = \mu c_1 \sinh(\mu x) + \mu c_2 \cosh(\mu x)$

Using $y(0) = 0$, $\quad 0 = c_1 \cdot 1 + c_2 \cdot 0 \Rightarrow c_1 = 0$

Using $y'(1) = 0$, $\quad 0 = \mu \cdot 0 \cdot \sinh(\mu) + \mu c_2 \cosh(\mu)$

$0 = \mu c_2 \sinh(\mu) \Rightarrow c_2 = 0$

$(\sinh(\mu) = 0 \Leftrightarrow \mu = 0)$

Thus $y(x) \equiv 0$.

Case II: $\lambda = 0$.

The ODE is $y'' = 0$. Then $y' = A$ and $y = Ax + B$.

Using $y(0) = 0$, $\quad 0 = A \cdot 0 + B \Rightarrow B = 0$.

Using $y'(1) = 0$, $\quad 0 = A \Rightarrow A = 0$. Only trivial soln.

Case III $\lambda > 0$. Let $\lambda = \mu^2$ $(\mu > 0)$.

Then the ODE $y'' + \mu^2 y = 0$. Char. eqn. $r^2 + \mu^2 = 0 \Rightarrow r = \pm \mu i$

solution $y(x) = c_1 \cos(\mu x) + c_2 \sin(\mu x)$

and $y'(x) = -\mu c_1 \sin(\mu x) + \mu c_2 \cos(\mu x)$

Using $y(0) = 0 \Rightarrow 0 = c_1 \cdot 1 + c_2 \cdot 0 \Rightarrow c_1 = 0$

Using $y'(1) = 0 \Rightarrow 0 = 0 + \mu c_2 \cos(\mu) \Rightarrow \cos(\mu) = 0$ $(c_2 \neq 0)$

$\mu = \frac{n}{2}\pi$, $n = 1, 3, 5, \ldots$

$\mu_n = \left(\frac{2n-1}{2}\right)\pi$, $n = 1, 2, 3, \ldots$

Thus eigenvalues, $\lambda_n = \mu_n^2 = \left(\frac{2n-1}{2}\right)^2 \pi^2$

eigenfunction (solution) $y_n(x) = c_2 \sin(\mu_n x)$

$$= \sin\left(\frac{2n-1}{2}\pi x\right), \quad n \geq 1$$

Normalize the eigenfunctions:

$$\phi_n = k_n \sin\left(\frac{2n-1}{2}\pi x\right).$$

Set

$$\int_0^1 k_n^2 \sin^2\left(\frac{2n-1}{2}\pi x\right) dx = 1$$

$\boxed{\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)}$

$$k_n^2 \int_0^1 \frac{1}{2}\left[1 - \cos(2n-1)\pi x\right] dx = 1$$

$$\frac{k_n^2}{2}\left[x - \frac{\sin(2n-1)\pi x}{(2n-1)\pi}\right]_0^1 = 1 \Rightarrow \frac{k_n^2}{2}\left[(1-0)-(0-0)\right] = 1$$

$$\Rightarrow k_n = \sqrt{2}$$

Thus $\phi_n(x) = \sqrt{2}\sin\left(\frac{2n-1}{2}\pi x\right), \quad n \geq 1$ --- (1)

Next find c_n's so that we have

$$x = f(x) = \sum_{n=1}^{\infty} c_n \phi_n(x). \Rightarrow c_n = \int_0^1 x\,\phi_n(x)\,dx$$

That is, $c_n = \int_0^1 x \cdot \sqrt{2}\sin\left(\frac{2n-1}{2}\pi x\right) dx$

$$= \sqrt{2}\int_0^1 x \cdot \sin\left(\frac{2n-1}{2}\pi x\right) dx$$

$$= (\text{IBP}) = \frac{4\sqrt{2}\,(-1)^{n+1}}{(2n-1)^2\pi^2}$$ --- (11)

So $b_n = \frac{c_n}{\lambda_n - \mu} = \frac{4\sqrt{2}\,(-1)^{n+1}}{(2n-1)^2\pi^2\left((2n-1)^2 \cdot \frac{\pi^2}{4} - 2\right)}$

Hence the solution is

$$y = \sum_{n=1}^{\infty} b_n \phi_n(x) = \sum_{n=1}^{\infty} \text{----} (11) \text{ and } (1).$$