Solve the differential equation $x^2y'' + 3xy' + (1+x)y = 0$ about $x = 0$.

Comparing the differential equation with $P(x)y''+Q(x)y'+R(x)y = 0$, we get $P(x) = x^2$. Note that $x = 0$ makes $P(x)$ zero, so $x = 0$ is a singular point.

Since $\lim_{x\to 0}(x-0)\dfrac{Q(x)}{P(x)} = \lim_{x\to 0} x\,\dfrac{3x}{x^2} = 3$ and $\lim_{x\to 0}(x-0)^2\,\dfrac{R(x)}{P(x)} = \lim_{x\to 0} x^2\,\dfrac{1+x}{x^2} = 1$ we conclude

that $x = 0$ is a regular singular point.

Let $y = x^r\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+r}$ be a solution.

Then $y' = \sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1}$, $y'' = \sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2}$.

Substituting these into the differential equation ($x^2y'' + 3xy' + y + xy = 0$), we get

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r} + 3\sum_{n=0}^{\infty}(n+r)a_n x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r+1} = 0$$

$$[r(r-1)+3r+1]a_0x^r + \sum_{n=1}^{\infty}\{[(n+r)(n+r-1)+3(n+r)+1]a_n + a_{n-1}\}x^{n+r} = 0$$

$$F(r)a_0x^r + \sum_{n=1}^{\infty}\{F(n+r)a_n + a_{n-1}\}x^{n+r} = 0, \quad \text{where } F(r) := r(r-1)+3r+1 = (r+1)^2$$

$$(a_0 \neq 0) \implies F(r) = 0, \quad \text{and} \quad F(n+r)a_n + a_{n-1} = 0, \quad n \geq 1$$

$$\implies r = -1, -1 \qquad a_n(r) = -\frac{a_{n-1}(r)}{F(n+r)}, \quad n \geq 1 \qquad \text{(Recurrence Relation)}$$

From the recurrence relation, we get

$$a_1 = -\frac{a_0}{F(r+1)} = -\frac{a_0}{(r+2)^2}$$

$$a_2 = -\frac{a_1}{F(r+2)} = \frac{a_0}{(r+3)^2(r+2)^2}$$

$$a_3 = -\frac{a_2}{F(r+3)} = -\frac{a_0}{(r+4)^2(r+3)^2(r+2)^2}$$

$$\vdots$$

$$a_n(r) = (-1)^n\frac{a_0}{(r+n+1)^2(r+n)^2\ldots(r+2)^2}, \quad n \geq 1$$

For the first solution ($@\, r=-1$), we get

$$a_n = (-1)^n \frac{1}{n^2 \cdot (n-1)^2 \ldots 2^2 \cdot 1^2} = (-1)^n \frac{1}{(n!)^2}, \qquad n \geq 0, \quad (a_0 = 1).$$

Thus the first solution is

$$y(x) = y_1(x) = \sum_{n=0}^{\infty} a_n\, x^{n-1} = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(n!)^2}\, x^{n-1}, \qquad x > 0.$$

Since $r_1 = r_2$, the second solution has the form

$$y = y_2(x) = y_1(x) \ln x + x^{r_1} \sum_{n=1}^{\infty} a_n'(r_1) x^n, \qquad x > 0.$$

To find $a_n'(r_1)$ we write $a_n(r) = (-1)^n \dfrac{a_0}{(r+n+1)^2 (r+n)^2 \ldots (r+2)^2}$, $n \geq 1$ as

$a_n(r) = (-1)^n \dfrac{1}{G_n(r)}$ taking $a_0 = 1$ where $G_n(r) := (r+n+1)^2 (r+n)^2 \ldots (r+2)^2$.

Note that $\dfrac{G_n'(r)}{G_n(r)} = \dfrac{2}{r+n+1} + \dfrac{2}{r+n} + \ldots + \dfrac{2}{r+2}$, using the fact that if

$f(x) = (x-a_1)^{b_1} (x-a_2)^{b_2} \ldots (x-a_n)^{b_n}$, then $\dfrac{f'(x)}{f(x)} = \dfrac{b_1}{x-a_1} + \dfrac{b_2}{x-a_2} + \ldots + \dfrac{b_n}{x-a_n}$.

Now

$$a_n(r_1) = \frac{d}{dr} a_n(r)]|_{r=r_1}$$

$$= \frac{d}{dr}\left[(-1)^n \frac{1}{G_n(r)}\right]_{r=-1} = (-1)^n \cdot -\frac{1}{G_n^2(r)} \cdot G_n'(r)|_{r=-1} = (-1)^{n+1} \frac{G_n'(r)}{G_n(r)} \cdot \frac{1}{G_n(r)}|_{r=-1}$$

$$= (-1)^{n+1} \left[\frac{2}{r+n+1} + \frac{2}{r+n} + \ldots + \frac{2}{r+2}\right] \left[\frac{1}{(r+n+1)^2 (r+n)^2 \ldots (r+2)^2}\right]_{r=-1}$$

$$= (-1)^{n+1} \left[\frac{2}{n} + \frac{2}{n-1} + \ldots + \frac{2}{1}\right] \left[\frac{1}{n^2 (n-1)^2 \ldots (1)^2}\right]$$

$$= (-1)^{n+1} 2H_n \frac{1}{(n!)^2}, \quad \text{where } H_n := 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}$$

Thus a second solution is

$$y_2(x) = y_1(x) \ln x + x^{-1} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2H_n}{(n!)^2}\, x^n, \qquad x > 0,$$

where y_1 is the first solution.