Solve: $xy'' + y' + y = 0$ about $x = 0$.

Soln: $P(x) = x = 0 \Rightarrow x = 0$ is a singular point.

Since $\lim_{x\to 0} x\, p(x) = \lim_{x\to 0} x \cdot \frac{1}{x} = 1$, $(= p_0)$

and $\lim_{x\to 0} x^2 q(x) = \lim_{x\to 0} x^2 \cdot \frac{1}{x} = 0$, $(= q_0)$

$x = 0$ is a regular singular point.

Indicial equation:

$$r(r-1) + p_0 r + q_0 = 0$$
$$r^2 - r + r + 0 = 0$$
$$r^2 = 0$$
$$\Rightarrow r = 0, 0.$$

For a first solution (take $r=0$):

Let $y = \sum_{n=0}^{\infty} a_n x^{n+r} = \sum_{n=0}^{\infty} a_n x^n$ be a solution. $(x>0)$

Then $y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$

$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$

Substitute these into the ode, we get

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-1} + \sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=1}^{\infty} (n+1)n\, a_{n+1} x^n + \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=1}^{\infty}(n+1)n\, a_{n+1}x^n + (0+1)a_1 x^0 + \sum_{n=1}^{\infty}(n+1)a_{n+1}x^n$$

$$+ a_0 x^0 + \sum_{n=1}^{\infty} a_n x^n = 0$$

$$(a_1+a_0) + \sum_{n=1}^{\infty}\left[(n+1)n\, a_{n+1} + (n+1)a_{n+1} + a_n\right]x^n = 0$$

$$(a_1+a_0) + \sum_{n=1}^{\infty}\left[(n+1)^2 a_{n+1} + a_n\right]x^n = 0$$

$$\Rightarrow \quad a_1 + a_0 = 0, \quad (n+1)^2 a_{n+1} + a_n = 0, \quad n \geq 1$$

$$\Rightarrow \quad a_1 = -a_0, \qquad a_{n+1} = -\frac{a_n}{(n+1)^2}, \quad n \geq 1$$

Now $\quad a_1 = -a_0$

$$a_2 = -\frac{a_1}{(2)^2} = \frac{a_0}{(2)^2}$$ (taking $n=1$ in the recurrence relation)

$$a_3 = -\frac{a_2}{(3)^2} = -\frac{a_0}{(3)^2(2)^2}$$

$$a_4 = -\frac{a_3}{(4)^2} = \frac{a_0}{(4)^2(3)^2(2)^2}$$

$$\vdots$$

$$a_n = \frac{(-1)^n a_0}{(n)^2(n-1)^2\cdots(2)^2}, \quad n \geq 1$$

$$= (-1)^n \frac{a_0}{(n!)^2}, \quad n \geq 0$$

Thus a solution is

(take $a_0 = 1$) $\quad y_1 = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{1}{(n!)^2} x^n.$

Since the roots of the indicial equation are equal, a second solution is given by

$$y = y_2(x) = y_1(x) \ln x + x^r \sum_{n=1}^{\infty} b_n x^n \quad , \quad x>0$$

$$= y_1 \ln x + \sum_{n=1}^{\infty} b_n x^n$$

$$= y_1 \ln x + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + \dots$$

Then

$$y' = y_1' \ln x + y_1 \cdot \frac{1}{x} + b_1 + 2b_2 x + 3b_3 x^2 + 4b_4 x^3 + 5b_5 x^4 + \dots$$

$$y'' = y_1'' \ln x + 2y_1' \cdot \frac{1}{x} + y_1 \cdot -\frac{1}{x^2} + 0 + 2b_2 + 6b_3 x + 12b_4 x^2 + \dots$$

$$= y_1'' \ln x + \frac{2y_1'}{x} - \frac{y_1}{x^2} + 2b_2 + 6b_3 x + 12b_4 x^2 + 20b_5 x^3 + \dots$$

Plug these into the ode

$$x y_1'' \ln x + 2y_1' - \cancel{\frac{y_1}{x}} + 2b_2 x + 6b_3 x^2 + 12b_4 x^3 + \dots$$
$$+ y_1' \ln x + \cancel{\frac{y_1}{x}} + b_1 + 2b_2 x + 3b_3 x^2 + 4b_4 x^3 + \dots$$
$$+ y_1 \ln x + b_1 x + b_2 x^2 + b_3 x^3 + \dots = 0$$

$$\ln x \, \cancelto{0}{[x y_1'' + y_1' + y_1]} + 2y_1' + b_1 + (2b_2 + 2b_2 + b_1)x +$$
$$(6b_3 + 3b_3 + b_2)x^2 + (12b_4 + 4b_4 + b_3)x^3$$
$$+ \dots = 0$$

or, $$b_1 + (4b_2 + b_1)x + (9b_3 + b_2)x^2$$
$$+ (16b_4 + b_3)x^3 + \dots = -2y_1' \quad \text{---}(*)$$

Let's work on the right hand side of (*):

$$-2y_1 = -2 \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{(n!)^2} = -2\left(1 - x + \frac{1}{4}x^2 - \frac{1}{36}x^3 + \frac{x^4}{24^2} + \dots\right.$$

so $$-2y_1' = -2\left(0 - 1 + \frac{1}{2}x - \frac{1}{12}x^2 + \frac{1}{144}x^3 + \dots\right) = 2 - x + \frac{1}{6}x^2 - \frac{x^3}{72}$$

$\longrightarrow$

Now from (*), equating coefficients, we get

$$b_1 = 2$$

$$4b_2 + b_1 = -1 \Rightarrow b_2 = -\frac{3}{4}$$

$$9b_3 + b_2 = \frac{1}{6} \Rightarrow b_3 = \frac{1}{9}\left(\frac{1}{6} + \frac{3}{4}\right) = \frac{11}{108}$$

$$16b_4 + b_3 = -\frac{1}{72} \Rightarrow b_4 = \frac{1}{16}\left(-\frac{1}{72} - \frac{11}{108}\right)$$

$$= \frac{1}{16} \cdot -\frac{25}{216} = -\frac{25}{3456}$$

Thus a second solution is

$$y_2(x) = y_1(x)\ln x + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + \cdots$$

$$= y_1(x)\ln x + 2x - \frac{3}{4}x^2 + \frac{11}{108}x^3 - \frac{25}{3456}x^4 + \cdots,$$

$$x > 0.$$