$$y'' + 2y' - 3y = 10e^{-3t} \quad \text{-----} \quad (*)$$

Consider $y'' + 2y' - 3y = 0$

Char. eqn: $r^2 + 2r - 3 = 0$ or, $(r+3)(r-1) = 0$

$\Rightarrow r = -3, 1$

$y_1(t) = e^{-3t}$, $y_2(t) = e^{t}$

So $y_c(t) = c_1 e^{-3t} + c_2 e^{t}$ (complementary solution)

Let $y = y_p(t) = Ate^{-3t}$ be a solution of the ODE $(*)$.

Then $y' = Ae^{-3t} - 3Ate^{-3t}$ (product rule)

$= Ae^{-3t} \cdot (1-3t)$

and $y'' = -3Ae^{-3t}(1-3t) + Ae^{-3t} \cdot -3$ (product rule)

$= -3Ae^{-3t}(1-3t+1)$

$= -3Ae^{-3t}(2-3t)$

Plug these into $(*)$, we get

$$-3Ae^{-3t}(2-3t) + 2Ae^{-3t}(1-3t) - 3Ate^{-3t} = 10e^{-3t}$$

$$A\cancel{e^{-3t}}\left[-3(2-3t) + 2(1-3t) - 3t\right] = 10\cancel{e^{-3t}}$$

$$A(-6 + \cancel{9t} + 2 - \cancel{6t} - \cancel{3t}) = 10$$

$$-4A = 10 \Rightarrow A = -\frac{5}{2}$$

So $y_p(t) = -\frac{5}{2}te^{-3t}$.

Hence $y(t) = y_c(t) + y_p(t)$

$= c_1 e^{-3t} + c_2 e^{t} - \frac{5}{2}te^{-3t}$.