Let x, y, and z denote the dimensions of the box (in meters). We would like to maximize the volume of the box

$$f(x, y, z) = xyz$$

subject to the constraint

$$g(x, y, z) = xy + 2xz + 2yz = 12.$$

The gradient vectors of f and g are

$$\nabla f(x, y, z) = \langle yz, xz, xy \rangle \quad \text{and} \quad \nabla g(x, y, z) = \langle y + 2z, x + 2z, 2x + 2y \rangle.$$

Consider the system

$$\begin{aligned} yz &= \lambda(y + 2z), \\ xz &= \lambda(x + 2z), \\ xy &= \lambda(2x + 2y), \\ xy + 2xz + 2yz &= 12. \end{aligned}$$

This is equivalent to the system

$$\begin{aligned} xyz &= \lambda(xy + 2xz), \\ xyz &= \lambda(xy + 2yz), \\ xyz &= \lambda(2xz + 2yz), \\ xy + 2xz + 2yz &= 12. \end{aligned}$$

Clearly $\lambda \neq 0$. (Otherwise, $x = y = z = 0$ which does not satisfy the constraint). The first two equations imply

$$\begin{aligned} xy + 2xz &= xy + 2yz \\ 2xz &= 2yz \\ x &= y. \end{aligned}$$

Similarly, the second and third equations yield

$$\begin{aligned} xy + 2yz &= 2xz + 2yz \\ xy &= 2xz \\ y &= 2z. \end{aligned}$$

Using the constraint,

$$xy + 2xz + 2yz = 4z^2 + 4z^2 + 4z^2 = 12z^2 = 12.$$

Then $z = 1$ which implies that $x = y = 2$. The maximum volume of the box is 4 cubic meters.