REFLEXIVITY, FACTORIZATION, AND HANKEL OPERATORS

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ABSTRACT. The space $\mathcal{H}ank$ of Hankel operators acting on the Hardy space $\mathbf{H}^2$ is a module over $\mathbf{H}^\infty$. There is a natural correspondence between weak* closed submodules of $\mathcal{H}ank$ and individual inner functions, and we apply work of V. Kapustin on Jordan models to characterize which submodules are reflexive in terms of the canonical factorization of these functions. We also prove that reflexivity of any weak* closed subspace of $\mathcal{H}ank$ is equivalent to reflexivity of the largest $\mathbf{H}^\infty$ module it contains. Analogous results are obtained in the finite dimensional and "semi-infinite" dimensional settings.

1. INTRODUCTION

In [7], A. Beurling set up a correspondence between invariant subspaces of the unilateral shift and inner functions. Applications in invariant subspace theory proper have been most successful for operators enjoying a symbolic calculus over the Hardy space $\mathbf{H}^\infty$. In [26], D. Sarason showed the algebra $\mathcal{AT}$ of *analytic* Toeplitz operators to be reflexive in the sense that any operator sharing its invariant subspaces must in fact belong to $\mathcal{AT}$. More recently, $\mathbf{H}^\infty$ functional calculi played a key role in Scott Brown's proof of the intransitivity of subnormal operators [8] and in joint work by H. Bercovici, S. Brown, and C. Pearcy establishing intransitivity of each operator of norm one whose spectrum contains the unit circle [3].

A. Loginov and V. Shulman, in [18], generalized the notion of reflexivity to operator spaces which are not necessarily closed under composition. The *reflexive closure* of a linear space $\mathbf{L}$ of Hilbert space operators, denoted $\operatorname{ref}\mathbf{L}$, consists of those operators B which belong to $\mathbf{L}$ locally in the sense that Bx belongs to the norm closure of $\mathbf{L}x$ for each vector x in the underlying Hilbert space. The space $\mathbf{L}$ is *reflexive* if it coincides with its reflexive closure; at the opposite extreme, $\mathbf{L}$ is *transitive* if every operator on the underlying Hilbert space belongs to its reflexive closure. The concept of reflexivity of subspaces played an important spoiler role in Warren Wogen's celebrated paper [28] in which he imbedded recalcitrant linear operators in singly–generated operator algebras, thereby solving several long standing questions concerning reflexivity of individual operators.

Hankel (*Toeplitz*) matrices are defined by the property that their entries only depend on the sum (respectively difference) of row and column number, i.e., they are "constant on cross-diagonals (diagonals)". The terms also refer to bounded Hilbert space operators whose matrices relative to "standard" orthonormal bases take this form. The best choice to make in the separable infinite-dimensional case is the Hardy space $\mathbf{H}^2$, equipped with its usual basis consisting of powers of z. Via the Nehari map, the space $\mathcal{H}ank$ of all Hankel operators on $\mathbf{H}^2$ is linearly isometrically isomorphic and weak* homeomorphic to $\mathbf{L}^\infty/(z\mathbf{H}^\infty)$, the Banach space dual of $\mathbf{H}^1$.

In [2], the first author and M. Ptak showed that each weak* closed linear space of (not necessarily analytic) Toeplitz operators is either reflexive or transitive and there are many of each type. By way of contrast, the first result of the present paper implies that no proper weak* closed subspace of $\mathcal{H}ank$ can be transitive.

Proposition 1.1. *$\mathcal{H}ank$ is transitive, but its subspaces are "relatively reflexive" in the sense that each weak* closed subspace $\mathbf{K}$ of $\mathcal{H}ank$ satisfies $\mathcal{H}ank \cap \operatorname{ref} \mathbf{K} = \mathbf{K}$.*

Both Toeplitz and Hankel operators have $\mathbf{L}^\infty$ symbols. Proposition 1.1 reflects the crucial difference: for Hankel operators, the Hilbert space inner product

$$\langle H_\phi f, g\rangle = \frac{1}{2\pi}\int \phi f g^*$$

involves the *analytic* function $g^* : z \mapsto \overline{g(\overline{z})}$. This means that studying reflexivity in $\mathcal{H}ank$ relies on cataloguing the different ways to factor an $\mathbf{H}^1$ function as a product of $\mathbf{H}^2$ functions, as opposed to the $\mathbf{H}^2 \cdot \overline{\mathbf{H}^2}$ factorizations which underlie [2].

We next take up reflexivity of subspaces of $\mathcal{H}ank$. Since reflexive closures are always weak* closed, all subspaces of $\mathcal{H}ank$ discussed below will be assumed to be closed in that topology. Given a subset N of $\mathbf{H}^1$, we define its *companion space* by

$$\mathcal{C}(N) := \left\{ H_\phi : \int \phi u = 0 \text{ for each } u \in N \right\}. \tag{1}$$

We will see that every weak* closed subspace of $\mathcal{H}ank$ is a companion space.

We will concentrate on two special companion spaces associated with an individual element u of $\mathbf{H}^1$: the companion space $\mathcal{C}_u$ of the singleton set $\{u\}$, and the companion space $\mathcal{M}_u$ of the set $u\mathbf{H}^\infty$. Every hyperspace of $\mathcal{H}ank$ takes the former form, and Beurling's Theorem will show that $\mathcal{M}_u$ represents the most general (weak* closed) submodule of $\mathcal{H}ank$.

Here is our main result. The proof, given in Section 5, does not depend on Section 4; it is accomplished by carefully studying a preadjoint of the Nehari map which sends trace class operators to $\mathbf{H}^1$ functions.

Theorem 1.2. *Let $N \subset \mathbf{H}^1$, take $\mathbf{W} := \mathcal{C}(N)$ to be the corresponding subspace of $\mathcal{H}ank$, and set θ as the greatest common divisor of the inner factors of the members of N. Then $\operatorname{ref} \mathbf{W} = \mathbf{W} + \operatorname{ref} \mathcal{M}_\theta$.*

To gauge the strength of this theorem, consider the special case when N consists of a single outer function ω and thus $\mathbf{W} = \mathcal{C}_\omega$. In this case $\theta = 1$ and $\mathcal{M}_\theta = \{0\}$ is trivially reflexive, hence the theorem above gives $\operatorname{ref} \mathcal{C}_\omega = \mathcal{C}_\omega$. So even though $\mathcal{C}_\omega$ is a hyperplane in the transitive space $\mathcal{H}ank$, the theorem still guarantees its reflexivity. More generally, Theorem 1.2 leads to equivalence of the first three conditions in the following result.

Theorem 1.3. *Let $N \subset \mathbf{H}^1$, take $\mathbf{W} := \mathcal{C}(N)$ to be the corresponding subspace of $\mathcal{H}ank$, and set θ as the greatest common divisor of the inner factors of the members of N. Then the following are equivalent:*

1. *$\mathbf{W}$ is reflexive.*
2. *$\mathcal{C}_\theta$ is reflexive.*
3. *$\mathcal{M}_\theta$ is reflexive.*
4. *$S(\theta)$ is reflexive.*
5. *$\theta = \beta\sigma$ where β is a Blaschke product having no repeated roots while σ is a singular inner function whose associated measure vanishes on Beurling-Carleson sets.*

In particular, reflexivity of $\mathbf{W}$ *is equivalent to reflexivity of the largest* $\mathbf{H}^\infty$ *module it contains.*

The equivalence (3) iff (4) is established in Section 4. In connection with Condition (4), recall that if θ is an inner function and S is the unilateral shift operator on $\mathbf{H}^2$ (i.e., multiplication by the independent variable z), the *Jordan model* operator $S(\theta)$ is defined to be the compression of S to the space $\mathbf{H}^2 \ominus \theta\mathbf{H}^2$. The equivalence of (4) and (5) is due to V. Kapustin [16, 17].

In Section 6, we determine the finite-rank members of ref $\mathcal{M}_\theta$. These are absent when θ is a totally nonatomic inner function, but plentiful in the opposite case that the measure associated with the singular factor of θ is supported on a countable set. This leads to a concrete computation of ref $\mathcal{M}_\theta$ in the latter situation.

All Hankel and Toeplitz operators are compressions of multiplication operators. So far, we have been discussing the so called "infinite" case of Hankel operators in $\mathbf{B}(\mathbf{H}^2)$. We now describe our results in the "semi-infinite" and "finite" cases, covered in the last two sections of the paper. (This less technical material can be read immediately after Section 2.) One of the original motivations of the present paper was the use of such settings to gain insight on the Toeplitz spaces studied in [2]. In fact, the simple device of reversing the order of a finite orthonormal basis shows reflexivity questions for spaces of Hankel and Toeplitz operators are entirely equivalent in these settings.

Write $\mathcal{P}_n$ for the set of all polynomials with degree less than or equal to n, thought of as a subspace of $\mathbf{H}^2$. Each of these spaces (including $\mathbf{H}^2$) is equipped with a standard orthonormal basis, and a bounded operator acting between two of them is considered to be *Hankel* if its matrix relative to the corresponding standard bases is constant on skew diagonals. We write $\mathcal{H}ank(\mathcal{P}_n, \mathbf{H}^2)$ for the space of Hankel operators in $\mathbf{B}(\mathcal{P}_n, \mathbf{H}^2)$ (the semi-infinite case) and $\mathcal{H}ank(\mathcal{P}_n, \mathcal{P}_m)$ for the Hankel operators in $\mathbf{B}(\mathcal{P}_n, \mathcal{P}_m)$ (the finite case). These operators have symbols as well, taken from $\overline{\mathbf{H}^2}$ in the semi-infinite case, and from $\overline{\mathcal{P}_{m+n}}$ in the finite case. Companion spaces are defined as in Display (1), with N ranging over the subsets of the corresponding preduals, $\mathbf{H}^2$ and $\mathcal{P}_{n+m}$ respectively.

Theorem 1.4. *Suppose* $n > 0$*. Let* $N \subset \mathbf{H}^2$*, take* $\mathbf{M} := \mathcal{C}(N)$ *to be the corresponding subspace of* $\mathcal{H}ank(\mathcal{P}_n, \mathbf{H}^2)$*, and set* γ *as the greatest common divisor of the inner factors of the members of* N*. Then the following are equivalent:*

(1) $\mathbf{M}$ *is reflexive.*
(2) $\mathcal{C}_\gamma$ *is reflexive.*
(3) γ *is a Blaschke product without repeated roots.*

The criteria in the next theorem are quite computable; one can check whether $r(z)$ has any repeated roots by applying the Euclidean algorithm to r and r'.

Theorem 1.5. *Let* N *be a set of polynomials in* $\mathcal{P}_{m+n}$*, take* $\mathbf{M} := \mathcal{C}(N)$ *to be the corresponding subspace of* $\mathcal{H}ank(\mathcal{P}_n, \mathcal{P}_m)$*, and set* $r = \gcd N$*. Then the following are equivalent:*

(1) $\mathbf{M}$ *is reflexive.*
(2) $\mathbf{M}$ *is contained in a reflexive hyperplane.*
(3) *Some member of* N *has degree* $\geq m + n - 1$ *and* $r(z)$ *has no repeated roots.*

Note the contrast between the middle conditions of the last two theorems: the hyperplane $\mathcal{C}_\gamma$ of 1.4 need not contain $\mathbf{M}$; on the other hand, reflexivity of $\mathbf{M}$ in

1.5 does not imply reflexivity of $\mathcal{C}_r$. Proofs techniques are also different: 1.4 takes advantage of the existence of a strictly cyclic vector for $\mathcal{H}ank(\mathcal{P}_n, \mathbf{H}^2)$; that tool is not available for 1.5 and we rely on a Theorem of Bertini instead.

2. Preliminaries

Hankel and Toeplitz Operators. We will concentrate on operators on the Hardy–Hilbert space $\mathbf{H}^2$ (see [12, 13, 15] for further information on $\mathbf{H}^2$), but we will use the letter $\mathcal{H}$ whenever we make a definition that is valid for general Hilbert space. We think of $\mathbf{H}^2$ both as a space of analytic functions and as a subspace of $\mathbf{L}^2 := \mathbf{L}^2(\mathbb{T})$. We also think of the space $\mathbf{L}^\infty$ as a subset of $\mathbf{L}^2$ and we define the subspace $\mathbf{H}^\infty$ of $\mathbf{L}^\infty$ as $\mathbf{L}^\infty \cap \mathbf{H}^2$. The dual space of $\mathbf{H}^1$ is $\mathbf{L}^\infty/(z\mathbf{H}^\infty)$ (see, for example, [12, p. 161]), under the pairing

$$\langle \phi + z\mathbf{H}^\infty, u\rangle := \frac{1}{2\pi}\int_{-\pi}^{\pi} \phi(e^{i\theta})u(e^{i\theta})\, d\theta,\ \ \phi + z\mathbf{H}^\infty \in \mathbf{L}^\infty/(z\mathbf{H}^\infty),\ \ u \in \mathbf{H}^1.$$

The (forward) shift operator S is defined on $\mathbf{H}^2$ as multiplication by the independent variable z. This has the effect of shifting the Fourier coefficients forward by one place (hence the name). Its adjoint, S^*, is usually called the backward shift since it shifts the Fourier coefficients one place back. The *flip operator* $J : \mathbf{L}^2 \to \mathbf{L}^2$, defined by $(Jf)(z) := f(\overline{z})$, interchanges analytic and coanalytic functions. If f is a function in $\mathbf{L}^2$, we also define f^* by $f^*(z) = \overline{f(\overline{z})}$; note that $f^* \in \mathbf{H}^2$ if and only $f \in \mathbf{H}^2$.

The main objects of study of this paper are Hankel operators. Most of the following basic facts concerning them can be found in [22, 23, 24].

We will say that an operator H is *Hankel* if, with respect to a canonical basis, the matrix of H has the property that all diagonals perpendicular to the main one (usually called "skew–diagonals") are constant.

The canonical basis for $\mathbf{H}^2$ is $\{z^n\}_{n=0}^\infty$, and thus $H \in \mathbf{B}(\mathbf{H}^2)$ is a Hankel operator if and only if it satisfies the equation $S^*H = HS$. We will denote the space of all bounded Hankel operators on $\mathbf{H}^2$ by $\mathcal{H}ank$.

Given $\phi \in \mathbf{L}^\infty$, define the operator $H_\phi : \mathbf{H}^2 \longrightarrow \mathbf{H}^2$ by

$$H_\phi f = PJ(\phi f),$$

where P denotes the orthogonal projection from $\mathbf{L}^2$ onto $\mathbf{H}^2$ and J is the flip operator defined above. Since $\phi \in \mathbf{L}^\infty$ we have that $H_\phi \in \mathbf{B}(\mathbf{H}^2)$; also $(H_\phi)^* = H_{\phi^*}$. It is easy to see that each such H_ϕ is a Hankel operator. The converse is a classical result of Nehari [20].

Theorem 2.1. *The Nehari map* $\Gamma^* : \mathbf{L}^\infty/(z\mathbf{H}^\infty) \to \mathbf{B}(\mathbf{H}^2)$*, defined by*

$$\Gamma^*(\phi + z\mathbf{H}^\infty) = H_\phi, \qquad \phi \in \mathbf{L}^\infty,$$

is an isometry with range $\mathcal{H}ank$*.*

Much of what we do in this article ultimately depends on the following basic computation.

Proposition 2.2. *Given a symbol* $\phi \in \mathbf{L}^\infty$ *and vectors* $f, g \in \mathbf{H}^2$*, we have*

$$\langle H_\phi f, g\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} \phi(e^{i\theta})f(e^{i\theta})g^*(e^{i\theta})d\theta.$$

Proof. By definition, we have $\langle H_\phi f, g\rangle = \langle PJ(\phi f), g\rangle = \langle J(\phi f), g\rangle$, which coincides with the integral $\frac{1}{2\pi}\int_{-\pi}^{\pi} \phi(e^{-it})f(e^{-it})\overline{g(e^{it})}dt$. The desired result follows by making the substitution $\theta = -t$ and recalling the definition of g^*. □

Recall that an operator $T \in \mathbf{B}(\mathbf{H}^2)$ is called a *Toeplitz* operator if $S^*TS = T$. This means that the matrix representation of T with respect to the canonical basis has the property that the diagonals parallel to the main one are constant.

For each $\phi \in \mathbf{L}^\infty$ define the operator $T_\phi : \mathbf{H}^2 \longrightarrow \mathbf{H}^2$ as

$$T_\phi f = P(\phi f),$$

where P is the orthogonal projection of $\mathbf{L}^2$ onto $\mathbf{H}^2$. Since $\phi \in \mathbf{L}^\infty$ we have that $T_\phi \in \mathbf{B}(\mathbf{H}^2)$. It is easy to see that the operator thus defined is a Toeplitz operator and one can show that all Toeplitz operators arise in this form.

The function ϕ above is called the *symbol* of the Toeplitz operator T_ϕ. Notice that in this case the symbols are unique. We shall need the fact that $T_\phi^* = T_{\overline{\phi}}$ for any $\phi \in \mathbf{L}^\infty$. When $\phi \in \mathbf{H}^\infty$, we will often write $\phi(S)$ for the Toeplitz operator T_ϕ and $\phi(S^*)$ for $T_{J(\phi)}$. These conventions, which are a special case of the $\mathbf{H}^\infty$ functional calculus for completely nonunitary contractions, will improve the appearance of many formulas in the sequel. (No corresponding convention is used for Hankel operators because powers of Hankel operators are usually not Hankel.) In particular, the shift and its adjoint are both examples of Toeplitz operators: $S = T_z$ and $S^* = T_{\overline{z}}$. For all these basic facts about Toeplitz operators, consult [12].

One way of thinking of the following well-known and easy to prove fact is that analytic Toeplitz operators implement the $\mathbf{H}^\infty$-module structure on $\mathcal{H}ank$. This makes the Nehari map a module isomorphism onto $\mathcal{H}ank$.

Proposition 2.3. *If $\psi \in \mathbf{L}^\infty$ and $\phi \in \mathbf{H}^\infty$ then $\phi(S^*)H_\psi = H_{\psi\phi} = H_\psi\phi(S)$.*

Duality. Assume that X is a Banach space, and that X^* is its dual space, where the duality is implemented by the bilinear form

$$\langle\cdot,\cdot\rangle : X^* \times X \longrightarrow \mathbb{C}.$$

In such a setting, the *annihilator* of a subset L of X is defined by

$$L^\perp := \{a \in X^* : \langle a, t\rangle = 0 \text{ for all } t \in L\},$$

while the *preannihilator* of a subset M of X^* is defined by

$$M_\perp := \{t \in X : \langle a, t\rangle = 0 \text{ for all } a \in M\}.$$

From topological vector space theory, we know that the map $L \mapsto L^\perp$ sets up a one-to-one correspondence between the collection of norm-closed linear manifolds in X and the collection of weak* closed linear manifolds in X^*; the inverse map is implemented by taking preannihilators. In fact, we will reserve the term *subspace* to refer to these two types of linear manifolds, i.e., either norm closed in X or weak* closed in X^*. Moreover, if M is a subset of X or X^*, we write $[M]$ for the subspace (closed in the appropriate topology) generated by M; given another subset $L \subset M$, we say L is *total* in M if $[L] = [M]$.

Reflexivity. Our main use of duality will involve the spaces $\mathbf{B}(\mathcal{H})$ and $\mathbf{Tr}(\mathcal{H})$, where $\mathbf{B}(\mathcal{H})$ is the full algebra of bounded linear operators acting on a separable Hilbert space $\mathcal{H}$, while $\mathbf{Tr}(\mathcal{H})$ is its ideal of trace class operators under the trace norm [27]. The space $\mathbf{B}(\mathcal{H})$ is the dual of $\mathbf{Tr}(\mathcal{H})$ under the pairing

$$\langle A, T\rangle := \mathrm{tr}\,(AT), \quad A \in \mathbf{B}(\mathcal{H}), \quad T \in \mathbf{Tr}(\mathcal{H}).$$

The corresponding weak* topology on $\mathbf{B}(\mathcal{H})$ is sometimes called the ultraweak operator topology.

The rank-one operators play an important role in what follows. Given $f, g \in \mathcal{H}$, the operator $f \otimes g$ is defined by $(f \otimes g)h = \langle h, g\rangle\, f$. Every operator of rank one or less takes this form, and each operator of rank n can be written as a sum of n such operators. The collection of operators having rank at most n is denoted $\mathbf{F}_n(\mathcal{H})$ and the collection of all operators of finite rank is denoted $\mathbf{F}(\mathcal{H})$; reference to the underlying Hilbert space is usually omitted. If A and B are bounded operators on $\mathcal{H}$, we have that $A(f \otimes g)B = (Af) \otimes (B^* g)$ and

$$\langle B, f \otimes g\rangle = \mathrm{tr}\,(B(f \otimes g)) = \langle Bf, g\rangle\,.$$

The first $\langle\cdot,\cdot\rangle$ in this equation reflects the duality between $\mathbf{B}(\mathcal{H})$ and $\mathbf{Tr}(\mathcal{H})$ (without adjunction), while the second $\langle\cdot,\cdot\rangle$ above refers to the usual inner product (involving conjugation) on $\mathcal{H}$.

Whenever we write a series of the form $\sum_{n=1}^{\infty} f_n \otimes g_n$, it will be assumed that $\sum \|f_n\|\|g_n\| < \infty$. Such series converge in the trace norm topology and every trace class operator can be represented in this way. The preceding display generalizes to

$$\left\langle B, \sum f_n \otimes g_n \right\rangle = \sum \langle Bf_n, g_n\rangle\,, \quad B \in \mathbf{B}(\mathcal{H}), \quad \sum f_n \otimes g_n \in \mathbf{Tr}(\mathcal{H}).$$

We now recall the following definitions, along with some of their basic properties. All of it can be found in [1]. Historical perspective can also be gained from [5], [11] (especially Section 22) and [25].

Definition 2.4. Let $\mathbf{L}$ be a linear manifold in $\mathbf{B}(\mathcal{H})$.

(1) The *reflexive closure* of $\mathbf{L}$ is given by

$$\mathrm{ref}\,\mathbf{L} \quad := \quad \left\{B \in \mathbf{B}(\mathcal{H}) : Bf \in \overline{\mathbf{L}f} \text{ for all } f \in \mathcal{H}\right\}$$

(2) The space $\mathbf{L}$ is *transitive* if $\mathrm{ref}\,\mathbf{L}$ exhausts the whole space of operators.

(3) The space $\mathbf{L}$ is *reflexive* if $\mathbf{L} = \mathrm{ref}\,\mathbf{L}$.

(4) A single operator $B \in \mathbf{B}(\mathcal{H})$ is *reflexive* if the weak* closed algebra generated by B and the identity operator is a reflexive linear manifold.

(5) We say $\mathbf{L}$ is *elementary* if $\mathbf{L}_\perp + \mathbf{F}_1$ exhausts the predual space $\mathbf{Tr}(\mathcal{H})$.

We always have $\mathbf{L} \subset \mathrm{ref}\,\mathbf{L} \subset \mathbf{B}(\mathcal{H})$, with (2) and (3) representing extreme possibilities. Note that $A \notin \mathrm{ref}\,\mathbf{L}$ if and only if there are vectors $f, g \in \mathcal{H}$ with $g \perp \mathbf{L}f$, but $\langle A, f \otimes g\rangle = \langle Af, g\rangle \neq 0$. This provides the useful characterization

$$\mathrm{ref}\,\mathbf{L} = (\mathbf{L}_\perp \cap \mathbf{F}_1)^\perp.$$

In particular, $\mathrm{ref}\,\mathbf{L}$ is always weak* closed and $\mathbf{L}$ is transitive if and only if $\mathbf{L}_\perp$ contains no rank-one operators; for weak* closed $\mathbf{L}$, reflexivity is tantamount to having $\mathbf{L}_\perp \cap \mathbf{F}_1$ total in $\mathbf{L}_\perp$.

More generally, for a positive integer k, a weak* closed subspace $\mathbf{L}$ is said to be *k-reflexive* if $\mathbf{L}_\perp \cap \mathbf{F}_k$ total in $\mathbf{L}_\perp$. There is a fleeting reference to 2-reflexivity in Theorem 3.2, but otherwise, this concept will not be needed in this paper.

Concept (5) above goes by the name Property $\aleph_1$ in [5]. We will find the third part of the following Proposition especially useful.

Proposition 2.5. *Let* $\mathbf{L}$ *be an elementary subspace of* $\mathbf{B}(\mathcal{H})$ *and suppose* $\mathbf{K}$ *is a subspace of* $\mathbf{L}$*.*

(1) $\mathbf{K} = \mathbf{L} \cap \operatorname{ref} \mathbf{K}$
(2) *No proper subspace of* $\mathbf{L}$ *can be transitive.*
(3) *If* $\operatorname{ref} \mathbf{K} \subset \mathbf{L}$ *then* $\mathbf{K}$ *is reflexive.*

Proof. Since every operator space is contained in its reflexive closure, we only have to be concerned with the inclusion $\supset$ of (1). So assume $a \notin \mathbf{K}$, but $a \in \mathbf{L}$. Then there is a trace class operator $t \in \mathbf{K}_\perp$ with $\langle a, t\rangle \neq 0$. Because $\mathbf{L}$ is elementary, we may even assume $t \in \mathbf{F}_1$, which means $a \notin \operatorname{ref} \mathbf{K}$, and (1) is established. (2) and (3) are immediate consequences of (1). □

We also need a few general facts concerning reflexivity.

Lemma 2.6. *Suppose* X, Y *are operators, while* $\mathcal{T}$ *is a linear space of operators.*

(1) *Then* $X(\operatorname{ref} \mathcal{T})Y \subset \operatorname{ref}(X\mathcal{T}Y)$*.*
(2) *If* $YXA = A$ *for each* $A \in \mathcal{T}$*, then*
   (a) $X\operatorname{ref} \mathcal{T} = \operatorname{ref}(X\mathcal{T})$*.*
   (b) $Y\operatorname{ref}(X\mathcal{T}) = \operatorname{ref} \mathcal{T}$*.*
   (c) $\mathcal{T}$ *is reflexive if and only if* $X\mathcal{T}$ *is reflexive.*

Proof. Given $C \in \operatorname{ref} \mathcal{T}$ and f in the underlying Hilbert space, we have

$$XCYf \in X[\overline{\mathcal{T}Yf}] \subset \overline{X\mathcal{T}Yf},$$

so $XCY \in \operatorname{ref}(X\mathcal{T}Y)$. Thus $X(\operatorname{ref} \mathcal{T})Y \subset \operatorname{ref}(X\mathcal{T}Y)$ and (1) is established.

In particular, $(I - XY)\operatorname{ref}(X\mathcal{T}) \subset \operatorname{ref}((X - XYX)\mathcal{T}) = \{0\}$, so

$$\operatorname{ref}(X\mathcal{T}) = XY\operatorname{ref}(X\mathcal{T}) \subset X\operatorname{ref}(YX\mathcal{T}) = X\operatorname{ref} \mathcal{T} \subset \operatorname{ref}(X\mathcal{T}),$$

and $\operatorname{ref}(X\mathcal{T}) = X\operatorname{ref} \mathcal{T}$, giving (2a). Multiplying the last equation by Y on the left also gives $Y\operatorname{ref}(X\mathcal{T}) = YX\operatorname{ref} \mathcal{T} = \operatorname{ref} \mathcal{T}$, and (2b) is established. Finally, (2c) follows immediately from (2a) and (2b). □

The reader primarily interested in the semi-infinite or finite cases should now skip to Section 7 or Section 8 respectively.

3. Cosymbols and companion spaces

The cosymbol map. The Nehari map Γ^* of Theorem 2.1 is weak* to weak* continuous. By topological vector space theory, we therefore know it is dual to a map $\Gamma : \mathbf{Tr}(\mathbf{H}^2) \to \mathbf{H}^1$. We refer to Γ as the *cosymbol map.* It will allow us to investigate various reflexivity questions in a function theoretic setting.

Proposition 3.1. *The cosymbol map* Γ *has the following properties.*

(1) Γ *is given by the explicit formula* $\Gamma(\sum f_n \otimes g_n) = \sum f_n g_n^*$*.*
(2) Γ *is a contraction.*
(3) Γ *is surjective. In fact, given* $u \in \mathbf{H}^1$*, there is an operator* $f \otimes g$ *of rank at most one satisfying* $\Gamma(f \otimes g) = u$ *and* $\|f \otimes g\|_{\mathbf{Tr}(\mathbf{H}^2)} = \|u\|_1$*.*
(4) *The kernel of* Γ *is* $\mathcal{H}ank_\perp$*.*

Proof. Given $f, g \in \mathbf{H}^2$ and $\phi \in \mathbf{L}^\infty$, we have

$$\langle \phi + z\mathbf{H}^\infty, \Gamma(f \otimes g)\rangle = \langle H_\phi, f \otimes g\rangle =< H_\phi f, g >=< \phi + z\mathbf{H}^\infty, fg^* >,$$

where the first equality comes from the definition of dual operator, while the third equality is the basic Proposition 2.2. This means $\Gamma(f \otimes g) = fg^*$, from which (1) follows by linearity and continuity. (2) is true since $\|\Gamma^*\| = 1$ by the Nehari Theorem.

It is well known that any function $u \in \mathbf{H}^1$ can be factored as $u = fg^*$ with $f,\ g \in \mathbf{H}^2$ and $\|f\|_2 = \|g\|_2 = \sqrt{\|u\|_1}$. It then follows that $\Gamma(f \otimes g) = u$ and $\|f \otimes g\|_{\mathbf{Tr}(\mathbf{H}^2)} = \|u\|_1$, which establishes (3).

Nehari's Theorem tells us that range$(\Gamma^*) = \mathcal{H}ank$, whence kernel$(\Gamma) = \mathcal{H}ank_\perp$ by general topological vector space theory. □

The full space $\mathcal{H}ank$. We now use Proposition 3.1 to obtain reflexivity properties of the full space of Hankel operators on $\mathbf{H}^2$. Theorem 3.2 was proven in [19].

Theorem 3.2. *The space $\mathcal{H}ank$ is transitive, but it is 2-reflexive and elementary.*

Proof. From Proposition 3.1, we know that $\mathcal{H}ank_\perp$ is the kernel of the cosymbol map Γ. In particular, $f \otimes g \in \mathcal{H}ank_\perp$ implies $fg^* = 0$, which only occurs when $f = 0$ or $g = 0$. Thus $\mathcal{H}ank_\perp$ has no rank-one members, which makes $\mathcal{H}ank$ transitive.

For 2-reflexivity, suppose $A \in (\mathcal{H}ank_\perp \cap \mathbf{F}_2)^\perp$. For each $f, g \in \mathbf{H}^2$, we have $\Gamma(f \otimes zg - zf \otimes g) = 0$, which implies $f \otimes zg - zf \otimes g \in \mathcal{H}ank_\perp \cap \mathbf{F}_2$ since $\ker \Gamma = \mathcal{H}ank_\perp$. Hence,

$$\langle (S^*A - AS)f, g\rangle = \langle A, f \otimes zg - zf \otimes g\rangle = 0,$$

implying $S^*A = AS$. Thus $A \in \mathcal{H}ank$.

To see that $\mathcal{H}ank$ is elementary, let $T \in \mathbf{Tr}(\mathbf{H}^2)$. Set $u := \Gamma(T)$, and choose $f, g \in \mathbf{H}^2$ as in Part (3) of Proposition 3.1. Then

$$T = (T - f \otimes g) + f \otimes g \in \mathcal{H}ank_\perp + \mathbf{F}_1.$$

Thus $\mathcal{H}ank_\perp + \mathbf{F}_1$ exhausts $\mathbf{Tr}(\mathbf{H}^2)$ and $\mathcal{H}ank$ is elementary by definition. (In fact we have shown that $\mathcal{H}ank_\perp + \mathrm{Ball}(\mathbf{F}_1)$ covers the unit ball of $\mathbf{Tr}(\mathbf{H}^2)$ which means that $\mathcal{H}ank$ has the stronger Property $\aleph_1(1)$ in the terminology of [5].) □

The following is a special case of Proposition 2.5. The third assertion will be used every time we show a subspace of $\mathcal{H}ank$ to be reflexive. Part (2) was obtained in [19].

Corollary 3.3. *Let $\mathbf{K}$ be a subspace of $\mathcal{H}ank$.*

1. $\mathbf{K} = \mathcal{H}ank \cap \mathrm{ref}\,\mathbf{K}$.
2. *No* proper *subspace of $\mathcal{H}ank$ can be transitive.*
3. ref $\mathbf{K} \subset \mathcal{H}ank$ *if and only if* $\mathbf{K}$ *is reflexive.*

Proof of Proposition 1.1: Observe that Proposition 1.1 is just the first assertion of Theorem 3.2 and Part (1) of Corollary 3.3. □

Companion spaces. Proposition 3.4 will enable us to move freely between subspaces of $\mathcal{H}ank$ and $\mathbf{H}^1$.

For notational clarity, we will use the symbol $\pm$ to denote annihilators (and preannihilators) relative to the duality between $\mathbf{L}^\infty/(z\mathbf{H}^\infty)$ and $\mathbf{H}^1$ mentioned at the beginning of the previous section while continuing to use the standard $\perp$ notation for annihilators relating to the duality between $\mathbf{B}(\mathbf{H}^2)$ and $\mathbf{Tr}(\mathbf{H}^2)$.

One further piece of notation will prove convenient for us. When M is a subspace of a Banach space X, we will write $\mathrm{LAT}(M)$ for the lattice of all subspaces of M and $\mathrm{COLAT}(M)$ for the lattice of all subspaces of X which contain M. The same conventions are adopted for (weak* closed) subspaces of X^*.

Proposition 3.4. *For each subset K of $\mathbf{H}^1$, we have $(\Gamma^{-1}(K))^\perp = \Gamma^*(K^\pm)$. Moreover, the following is a commutative diagram of lattice isomorphisms and anti-isomorphisms.*

$$\begin{array}{ccc} \mathrm{LAT}(\mathbf{H}^1) & \xrightarrow{\pm} & \mathrm{LAT}(\mathbf{L}^\infty/(z\mathbf{H}^\infty)) \\ {\scriptstyle \Gamma^{-1}}\big\downarrow & & \big\downarrow{\scriptstyle \Gamma^*} \\ \mathrm{COLAT}(\mathcal{H}ank_\perp) & \xrightarrow{\perp} & \mathrm{LAT}(\mathcal{H}ank) \end{array}$$

Proof. Given $\phi + z\mathbf{H}^\infty \in K^\pm$ and $t \in \Gamma^{-1}(K)$, we have $\langle \Gamma^*(\phi + z\mathbf{H}^\infty), t\rangle = \langle \phi + z\mathbf{H}^\infty, \Gamma(t)\rangle = 0$, so $(\Gamma^{-1}(K))^\perp \supset \Gamma^*(K^\pm)$. To get the opposite inclusion, take $A \in (\Gamma^{-1}(K))^\perp$. Since $\Gamma^{-1}(K) \supset \operatorname{kernel}\Gamma$, we see that $A \in (\mathcal{H}ank_\perp)^\perp = \mathcal{H}ank$. Since the range of the Nehari map Γ^* exhausts $\mathcal{H}ank$, we can write $A = \Gamma^*(\phi + z\mathbf{H}^\infty)$ for some $\phi \in \mathbf{L}^\infty$. Let $f \in K$. By surjectivity of Γ, we have $f = \Gamma(t)$ for some $t \in \Gamma^{-1}(K)$, and

$$< \phi + z\mathbf{H}^\infty, f >=< A, t >= 0.$$

This shows that $\phi + z\mathbf{H}^\infty \in K^\pm$, whence $A \in \Gamma^*(K^\pm)$ as desired.

Since any $M \in \mathrm{LAT}(\mathbf{L}^\infty/z\mathbf{H}^\infty)$ can be expressed as $M = K^\pm$ for $K = M_\pm$, we see that Γ^* maps $\mathrm{LAT}(\mathbf{L}^\infty/z\mathbf{H}^\infty)$ into $\mathrm{LAT}(\mathcal{H}ank)$, so all the maps in the diagram are well-defined, and the diagram commutes.

To complete the proof, it suffices to show that three of the lattice maps in the diagram are bijective. Like all annihilator maps, the horizontal maps $\pm, \perp$ are lattice anti-isomorphisms. Write π for the natural projection map from $\mathbf{Tr}(\mathbf{H}^2)$ to $\mathbf{Tr}(\mathbf{H}^2)/\mathcal{H}ank_\perp$. Then π induces a lattice isomorphism between $\mathrm{COLAT}(\mathcal{H}ank_\perp)$ and $\mathrm{LAT}(\mathbf{Tr}(\mathbf{H}^2)/\mathcal{H}ank_\perp)$. Also, $\Gamma \circ \pi^{-1}$ is a Banach space isomorphism and hence sets up a lattice isomorphism between $\mathrm{LAT}(\mathbf{Tr}(\mathbf{H}^2)/\mathcal{H}ank_\perp)$ and $\mathrm{LAT}(\mathbf{H}^2)$. Thus $\Gamma = (\Gamma \circ \pi^{-1}) \circ \pi$ provides a lattice isomorphism, and the vertical map Γ^{-1} is our third lattice bijection. □

Definition 3.5. The *companion space* to a sub*set* K of $\mathbf{H}^1$ is defined as

$$\mathcal{C}(K) := \{H_\phi \in \mathcal{H}ank : \int \phi u = 0 \ \text{ for each } \ u \in K\}.$$

For an individual $u \in \mathbf{H}^1$, we write $\mathcal{C}_u := \mathcal{C}(\{u\})$ and $\mathcal{M}_u := \mathcal{C}(u\mathbf{H}^\infty)$.

Thus, in duality notation, we have $\mathcal{C}(K) = \Gamma^*(K^\pm)$, which coincides with $(\Gamma^{-1}(K))^\perp$ by the last proposition.

Here are some observations on companion spaces.

Proposition 3.6. *Let $u \in \mathbf{H}^1$ and $K \subset \mathbf{H}^1$.*

(1) *$\mathcal{C}(K)$ only depends on the closed linear span $[K]$ of K.*
(2) *$\mathcal{C}$ sets up a lattice anti-isomorphism between the subspaces of $\mathbf{H}^1$ and subspaces of $\mathcal{H}ank$.*
(3) *$\mathcal{C}_u$ is the most general hyperspace of $\mathcal{H}ank$.*
(4) *$\mathcal{M}_u$ is the most general submodule of $\mathcal{H}ank$; it only depends on the inner factor of u.*
(5) *Let s be the greatest common divisor of all inner factors of members of $[K]$. Then $\mathcal{M}_s$ is the largest submodule of $\mathcal{C}(K)$.*
(6) *In order for a rank-one operator $f \otimes g$ to belong to $[\mathcal{C}(K)]_\perp$ it is necessary and sufficient that $fg^* \in [K]$.*

Proof. We will be referring to the diagram of Proposition 3.4.

(1) We have $\mathcal{C}(K) = \Gamma^*(K^\pm)$ and $K^\pm = [K]^\pm$.
(2) This follows from the bijectivity of the lattice maps in the diagram.
(3) The Nehari map Γ^* preserves codimension (since it is a vector space isomorphism), while taking annihilators interchanges codimension with dimension.
(4) Given $\phi + z\mathbf{H}^\infty \in \mathbf{L}^\infty/(z\mathbf{H}^\infty)$, $u \in \mathbf{H}^1$, and $f \in \mathbf{H}^\infty$, we have

$$\langle f(\phi + z\mathbf{H}^\infty), u\rangle = \langle \phi + z\mathbf{H}^\infty, fu\rangle .$$

It follows that $N^\pm$ is a submodule of $\mathbf{L}^\infty/(z\mathbf{H}^\infty)$ whenever N is a submodule of $\mathbf{H}^1$; conversely, when K is a submodule of $\mathbf{L}^\infty/(z\mathbf{H}^\infty)$, its preannihilator $K_\pm$ must be a submodule of $\mathbf{H}^1$. Notice that Γ^* is an $\mathbf{H}^\infty$-module isomorphism and thus M is a submodule of $\mathcal{H}ank$ if and only if $\Gamma^{*-1}(M)$ is a submodule of $\mathbf{L}^\infty/(z\mathbf{H}^\infty)$, and this happens if and only if $(\Gamma^{*-1}(M))_\pm$ is a submodule of $\mathbf{H}^1$. But Beurling's Theorem tells us that every submodule of $\mathbf{H}^1$ is singly-generated and of the form $[u\mathbf{H}^\infty]$. Also, it only depends on the inner factor of u. Thus $(\Gamma^{*-1}(M))_\pm = [u\mathbf{H}^\infty]$ for some $u \in \mathbf{H}^1$ and since $\mathcal{M}_u = \mathcal{C}(u\mathbf{H}^\infty) = \Gamma^*([u\mathbf{H}^\infty]^\pm)$, it follows that M must equal $\mathcal{M}_u$.
(5) The largest submodule M of $\mathcal{C}(K)$ is $\{H_\phi : H_{f\phi} \in \mathcal{C}(K) \text{ for all } f \in \mathbf{H}^\infty\}$. The submodule $\mathcal{C}^{-1}(M)$ equals $[\mathbf{H}^\infty K]$ which by Beurling's Theorem equals $s\mathbf{H}^\infty$.
(6) By Part (1) we know that $\mathcal{C}(K) = \mathcal{C}([K]) = (\Gamma^{-1}([K]))^\perp$, so $[\mathcal{C}(K)]_\perp = \Gamma^{-1}([K])$ by continuity. But $f \otimes g$ belongs to the latter space if and only if $fg^* = \Gamma(f \otimes g) \in [K]$.

□

Parts (4) and (5) of the last proposition motivate the following definition.

Definition 3.7. Let M be a subspace of $\mathcal{H}ank$. The greatest common divisor of all inner factors of members of $\mathcal{C}^{-1}(M)$ is called the *seed* of M and denoted seed (M).

Corollary 3.8. *Let $\mathbf{W}$ be a subspace of $\mathcal{H}ank$ with seed θ.*

(1) *$\mathcal{M}_\theta$ is the largest $\mathbf{H}^\infty$-submodule of $\mathcal{H}ank$ contained in $\mathbf{W}$.*
(2) *The rank-one operator $f \otimes g$ belongs to $\mathbf{W}_\perp$ if and only if $fg^* \in \mathcal{C}^{-1}(\mathbf{W})$.*
(3) *The rank-one operator $f \otimes g$ belongs to $[\mathcal{M}_\theta]_\perp$ if and only if θ divides fg^*.*

Proof. Part (1) follows by applying Proposition 3.6(5) with $K := \mathcal{C}^{-1}(\mathbf{W})$.

To obtain part (2), apply Proposition 3.6(6) to $K := \mathcal{C}^{-1}(\mathbf{W})$.

For (3), apply Proposition 3.6(6) to $K := \theta\mathbf{H}^\infty$ to see that $f \otimes g \in [\mathcal{M}_\theta]_\perp$ if and only if $fg^* \in [\theta\mathbf{H}^\infty]$. But the latter condition is equivalent to fg^* being divisible by θ. □

4. Reflexivity of submodules of $\mathcal{H}ank$

In Theorem 4.5, we establish the equivalence of Conditions (3) and (4) of Theorem 1.3, which we then combine with Kapustin's result to the effect that Conditions (4) and (5) of Theorem 1.3 are also equivalent. Theorem 4.5 is a two-way street which can be used to apply the results of Section 6 below to Jordan models.

Definition 4.1. Let θ be an inner function. The subspace $\mathbf{H}^2 \ominus \theta\mathbf{H}^2$ is denoted by $\mathcal{H}(\theta)$. The *Jordan model* $S(\theta) : \mathbf{H}^2 \longrightarrow \mathbf{H}^2$ is defined by

$$S(\theta) := P_{\mathcal{H}(\theta)}S.$$

Here $P_{\mathcal{H}(\theta)}$ is the orthogonal projection from $\mathbf{H}^2$ to $\mathcal{H}(\theta)$.

Note that since $\theta\mathbf{H}^2$ is invariant under the shift S, we actually have $S(\theta) = P_{\mathcal{H}(\theta)}S = P_{\mathcal{H}(\theta)}SP_{\mathcal{H}(\theta)}$. In fact, the usual convention is to restrict the domain and codomain of $S(\theta)$ to $\mathcal{H}(\theta)$ (the resulting operator being called the *compression of the shift S to $\mathcal{H}(\theta)$*) but it simplifies our notation to have all these operators acting on the same Hilbert space $\mathbf{H}^2$. Jordan models have been studied extensively (see, for example, [4, 21]) and it is well known that they are related to Hankel operators. There is a natural $\mathbf{H}^\infty$ functional calculus for $S(\theta)$. Proofs of Theorem 4.3 can be found, for example, in [4, p. 41] and [21, p. 230].

Definition 4.2. For every $\phi \in \mathbf{H}^\infty$, the operator $\phi(S(\theta))$ on $\mathbf{H}^2$ is defined by $\phi(S(\theta)) := P_{\mathcal{H}(\theta)}\phi(S)$.

Theorem 4.3. *Let θ be an inner function. Then the weak* closed algebra generated by $S(\theta)$ is*

$$\mathcal{A}_{S(\theta)} := \{\phi(S(\theta)) \, : \, \phi \in \mathbf{H}^\infty\}.$$

The identity of this algebra is $P_{\mathcal{H}(\theta)}$; when domains and codomains are restricted to $\mathcal{H}(\theta)$, the algebra $\mathcal{A}_{S(\theta)}$ becomes the commutant of $S(\theta)$.

The following variant of the Commutant Lifting Theorem (see e.g. [21, p. 230]) provides our basic tool for relating spaces of Jordan models and Hankel operators.

Lemma 4.4. *Let θ be an inner function and write $X := H_{z\overline{\theta}}$.*

(1) *$X^*X = P_{\mathcal{H}(\theta)}$, i.e., X is a partial isometry with initial space $\mathcal{H}(\theta)$.*
(2) *For every function $\phi \in \mathbf{H}^\infty$ we have $H_{z\overline{\theta}\phi} = X\phi(S(\theta))$.*
(3) *$\mathcal{M}_\theta = X\mathcal{A}_{S(\theta)}$.*

Proof. (1) can be found on Page 34 of [24]; it can also be checked directly.

$H_{z\overline{\theta}}\phi(S) = H_{z\overline{\theta}\phi}$ by Proposition 2.3. Since $H_{z\overline{\theta}} = H_{z\overline{\theta}}P_{\mathcal{H}(\theta)}$ by Part (1), we have

$$H_{z\overline{\theta}\phi} = H_{z\overline{\theta}}\phi(S) = H_{z\overline{\theta}}P_{\mathcal{H}(\theta)}\phi(S) = H_{z\overline{\theta}}\phi(S(\theta)),$$

establishing (2).

For (3), first apply the definition to note that the Hankel operator H_ψ is in $\mathcal{M}_\theta$ if and only if $\int \psi\theta f = 0$ for all $f \in \mathbf{H}^\infty$, i.e., if and only if $\psi\theta \in z\mathbf{H}^\infty$ which is in turn equivalent to ψ being of the form $z\overline{\theta}\phi$ for some function $\phi \in \mathbf{H}^\infty$.

That is, a Hankel operator is in $\mathcal{M}_\theta$ if and only if it has the form $H_{z\overline{\theta}\phi}$ for $\phi \in \mathbf{H}^\infty$. As ϕ ranges through $\mathbf{H}^\infty$, Part (2) implies that the corresponding Hankel operators $H_{z\overline{\theta}\phi}$ exhaust $\mathcal{M}_\theta$, while $\phi(S(\theta))$ traces out $\mathcal{A}_{S(\theta)}$ by Theorem 4.3. ☐

Theorem 4.5. *Let θ be an inner function. Then:*

(1) $\operatorname{ref}\mathcal{M}_\theta = H_{z\overline{\theta}}\operatorname{ref}\mathcal{A}_{S(\theta)}$.
(2) $\operatorname{ref}\mathcal{A}_{S(\theta)} = (H_{z\overline{\theta}})^*\operatorname{ref}\mathcal{M}_\theta$.
(3) *$\mathcal{M}_\theta$ is a reflexive module of Hankel operators if and only if $S(\theta)$ is a reflexive Jordan model.*

Proof. Take $X = H_{z\overline{\theta}}$ and $Y = X^*$, and write $\mathcal{T} := \mathcal{A}_{S(\theta)}$. The preceding lemma shows that the hypothesis of Lemma 2.6(2) is satisfied and hence the conclusions of the current Theorem follow. □

Kapustin's Theorem. We close this section with some detail concerning the equivalence of parts (4) and (5) of Theorem 1.3.

Definition 4.6. Let E be a closed subset of $(-\pi, \pi]$. We say that E is a *Beurling-Carleson set* (a BC-set) if E has zero Lebesgue measure and the sum $\sum \ell_k \log \frac{2\pi}{\ell_k}$ is finite. Here $\{\ell_k\}$ consists of the lengths of the intervals making up the complement of E.

We say that a measure μ is BC-vanishing if $\mu(E) = 0$ for every BC-set E.

Note in particular that singleton point sets are BC and thus BC-vanishing measures must be totally nonatomic.

Here is the main result of [16, 17]

Theorem 4.7 (Kapustin)**.** *Let θ be an inner function and write μ for the representing measure of the singular factor of θ. The following statements are equivalent.*

(1) *The function θ has no multiple zeros in $\mathbb{D}$ and $\mu(E) = 0$ for every BC-set E.*
(2) *The algebra $\mathbf{H}^\infty/\theta\mathbf{H}^\infty$ is the weak*-closed linear span of its idempotent members.*
(3) *The Jordan model $S(\theta)$ is reflexive.*

Regarding the proof of Theorem 1.3: That (3) $\iff$ (4) is the content of Theorem 4.5, while (4) $\iff$ (5) follows from Theorem 4.7. □

5. Reflexivity of subspaces of $\mathcal{H}ank$

In this section, we complete the proofs of Theorems 1.2 and 1.3. When ω is outer, $\mathcal{M}_\omega = \{0\}$ is the only submodule of $\mathcal{C}_\omega$. The 0-module is trivially reflexive, so our first job is establishing reflexivity of the hyperspace $\mathcal{C}_\omega$.

Proposition 5.1. *If ω is outer then $\mathcal{C}_\omega$ is reflexive.*

Proof. Let $B \in \operatorname{ref}\mathcal{C}_\omega = ((\mathcal{C}_\omega)_\perp \cap \mathbf{F}_1)^\perp$. Factor $\omega = \omega_1\omega_2^*$ as the product of two outer $\mathbf{H}^2$ functions. Given a real number $a \in (-1, 1)$, define functions $f, g \in \mathbf{H}^2$ by $f(z) := 1 - az$ and $g(z) := \frac{1}{1-az}$. Then $fg = 1$ so $(g\omega_1)(f^*\omega_2)^* = \omega$. In view of Corollary 3.8(2), we have $(g\omega_1) \otimes (f^*\omega_2) \in (\mathcal{C}_\omega)_\perp$ and hence

$$\big\langle (I - aS^*)B(I - aS)^{-1}\omega_1, \omega_2\big\rangle = \langle Bg\omega_1, f^*\omega_2\rangle = 0, \quad -1 < a < 1.$$

Expanding this expression as a power series in a yields

$$\left(B + \sum_{n=0}^{\infty} a^{n+1}(BS - S^*B)S^n\right)\omega_1 \perp \omega_2, \quad -1 < a < 1.$$

Since the orthogonal complement of ω_2 is linear and (norm) closed, repeatedly differentiating and evaluating at $a = 0$ shows the coefficient of each power of a must belong to this space as well:

$$\langle (BS - S^*B)S^n\omega_1, \omega_2\rangle = 0 \quad n = 0, 1, \ldots.$$

Replacing f by $f(z) := (1 - az)\exp(bz^m)$ and g by $g(z) := \frac{\exp(-bz^m)}{1-az}$ in the above argument, the last display becomes

$$\langle \exp(b(S^*)^m)(BS - S^*B)S^n \exp(-bS^m)\omega_1, \omega_2\rangle = 0, \quad -1 < b < 1;\ m, n \in \mathbb{N}.$$

Differentiating the last expression once with respect to b and then setting $b = 0$ yields

$$\langle (S^*)^m(BS - S^*B)S^n\omega_1, \omega_2\rangle - \big\langle (BS - S^*B)S^{m+n}(\omega_1), \omega_2\big\rangle = 0.$$

Since we already know the second inner product in this display is zero, we conclude that

$$(BS - S^*B)z^n\omega_1 \perp z^m\omega_2, \quad m, n \in \mathbb{N}.$$

Since ω_1, ω_2 are outer, this means $BS - S^*B = 0$, so $B \in \mathcal{H}ank$. An appeal to Corollary 3.3(3) completes the proof. □

Our next task is to show more generally that reflexivity of $\mathcal{C}_{\theta\omega}$ does not depend on the outer factor ω. The plan is to use Toeplitz operators to build up the class of inner functions θ for which this is true. Proposition 5.1 got us started with the case $\theta = 1$.

Proposition 5.2. *Let $u \in \mathbf{H}^1$ and ϕ, $\psi \in \mathbf{H}^\infty$.*

(1) *If $\psi\phi$ divides u, then $B \in \operatorname{ref}\mathcal{C}_u$ implies $\phi(S^*)B\psi(S) \in \operatorname{ref}\mathcal{C}_{u/(\psi\phi)}$.*
(2) *If $\psi\phi$ divides u, then $B \in \operatorname{ref}\mathcal{M}_u$ implies $\phi(S^*)B\psi(S) \in \operatorname{ref}\mathcal{M}_{u/(\psi\phi)}$.*
(3) *$B \in \operatorname{ref}\mathcal{M}_u$ implies $\phi(S^*)B\psi(S) \in \operatorname{ref}\mathcal{M}_u$.*

Proof. Given $H_\eta \in \mathcal{C}_u$, we have $\int \eta u = 0$ by definition. Proposition 2.3 yields $\phi(S^*)H_\eta\psi(S) = H_{\phi\eta\psi}$. Since we already know $\int(\phi\eta\psi)(u/(\psi\phi)) = 0$, we conclude that $\phi(S^*)\mathcal{C}_u\psi(S) \subset \mathcal{C}_{u/(\psi\phi)}$ Applying Lemma 2.6(1), we then get

$$\phi(S^*)(\operatorname{ref}\mathcal{C}_u)\psi(S) \subset \operatorname{ref}(\phi(S^*)\mathcal{C}_u\psi(S)) \subset \operatorname{ref}(\mathcal{C}_{u/(\psi\phi)}),$$

which establishes the first assertion of the proposition. The second assertion can be proved in the same manner. The third assertion follows from the fact that, since $\mathcal{M}_u$ is an $\mathbf{H}^\infty$-module, $\phi(S^*)\mathcal{M}_u\psi(S) \subset \mathcal{M}_u$. Now apply Lemma 2.6(1) as above. □

Corollary 5.3. *Suppose $\mathcal{F}$ is a family of inner functions having a least common multiple v.*

(1) *$\operatorname{ref}\mathcal{M}_v$ is the (weak*) closed linear span of $\{\operatorname{ref}\mathcal{M}_u : u \in \mathcal{F}\}$.*
(2) *In order that $\mathcal{M}_v$ be reflexive it is necessary and sufficient that $\mathcal{M}_u$ be reflexive for each $u \in \mathcal{F}$.*
(3) *In order that $\mathcal{C}_v$ be reflexive it is sufficient that $\mathcal{C}_u$ be reflexive for each $u \in \mathcal{F}$.*

Proof. Write $\mathbf{W}$ for the (weak*) closed linear span of $\{\operatorname{ref}\mathcal{M}_u : u \in \mathcal{F}\}$. Since $\mathcal{M}_v \supset \mathcal{M}_u$ for each $u \in \mathcal{F}$, we at least have $\mathbf{W} \subset \operatorname{ref}\mathcal{M}_v$ of (1). To get the opposite inclusion, let $B \in \operatorname{ref}\mathcal{M}_v$. Then for each $u \in \mathcal{F}$ and each $p \in \frac{v}{u}\mathbf{H}^\infty$, $Bp(S) \in \operatorname{ref}\mathcal{M}_u \subset \mathbf{W}$ by parts (3) and (2) of the previous proposition. Since

$\gcd\{\frac{v}{u} : u \in \mathcal{F}\} = 1$, we conclude by Beurling's Theorem that the constant function 1 belongs to the (weak*) closed linear span of $\{\frac{v}{u}\mathbf{H}^\infty : u \in \mathcal{F}\}$, whence $B \in \mathbf{W}$ since $\mathbf{W}$ is a (weak*) closed subspace. This establishes (1).

To prove (2) observe first that, since $\mathcal{M}_v \supset \mathcal{M}_u$, reflexivity of $\mathcal{M}_v$ implies reflexivity of $\mathcal{M}_u$ for all $u \in \mathcal{F}$, by Corollary 3.3(3). For the opposite direction, observe that since $\mathcal{M}_v$ is the (weak*) closed linear span of $\{\mathcal{M}_u : u \in \mathcal{F}\}$, reflexivity of $\mathcal{M}_v$ follows from part (1).

For (3), suppose each $\mathcal{C}_u$ is reflexive and let $B \in \operatorname{ref} \mathcal{C}_v$. Then the previous proposition yields $B\left(\frac{v}{u}\right)(S) \in \operatorname{ref} \mathcal{C}_u \subset \mathcal{H}ank$ for each $u \in \mathcal{F}$. Define

$$\mathcal{I} := \{\psi \in \mathbf{H}^\infty : B\psi(S) \in \mathcal{H}ank\},$$

which is a (weak*) closed ideal. It then follows that $\frac{v}{u} \in \mathcal{I}$ for all $u \in \mathcal{F}$. Since $\gcd\{\frac{v}{u} : u \in \mathcal{F}\} = 1$, Beurling's Theorem tells us the constant function 1 is in $\mathcal{I}$, whence $B \in \mathcal{H}ank$. □

The span of any two distinct hyperplanes in $\mathcal{H}ank$ will contain all of $\mathcal{H}ank$ so (1) fails rather spectacularly when $\mathcal{M}$ is replaced by $\mathcal{C}$. However, the converse of (3) is an immediate consequence of (2) and Corollary 5.11 below. Similar comments apply to the following dual of the last Corollary.

Corollary 5.4. *Suppose $\mathcal{F}$ is a family of inner functions and write v for its greatest common divisor. Then* $\operatorname{ref} \mathcal{M}_v$ *is the intersection of* $\{\operatorname{ref} \mathcal{M}_u : u \in \mathcal{F}\}$.

Proof. Write $\mathbf{W}$ for the intersection of $\{\operatorname{ref} \mathcal{M}_u : u \in \mathcal{F}\}$. Since $\mathcal{M}_v \subset \mathcal{M}_u$ for each $u \in \mathcal{F}$, we at least have $\operatorname{ref} \mathcal{M}_v \subset \mathbf{W}$. To get the opposite inclusion, let $B \in \operatorname{ref} \mathbf{W}$. Then for each $u \in \mathcal{F}$, we have $B\frac{u}{v}(S) \in \frac{u}{v}(S)\operatorname{ref} \mathcal{M}_u \subset \operatorname{ref} \mathcal{M}_v$. Define

$$\mathcal{I} := \{\psi \in \mathbf{H}^\infty : B\psi(S) \in \operatorname{ref} \mathcal{M}_v\},$$

which is a (weak*) closed ideal. Since $\gcd\{\frac{u}{v} : u \in \mathcal{F}\} = 1$, Beurling's Theorem allows us to conclude that $B \in \operatorname{ref} \mathcal{M}_v$ as well. □

Consistency. Recall that, by Proposition 2.3 above, if it is known a priori that $A \in \mathcal{H}ank$, then $\phi(S^*)A$ and $A\phi(S)$ are both Hankel and they agree. If $A \notin \mathcal{H}ank$, it still might turn out that one of $\phi(S^*)A$ or $A\phi(S)$ is Hankel, but this doesn't guarantee that they're both Hankel and even when they are, they might not agree. For simple examples, take $\phi(z) = z$ with $A = 1 \otimes z^2$ and $A = 1 \otimes z$. The goal of the next three lemmas is to show that this pathology doesn't arise when we apply Proposition 5.2.

The rank-one Hankel operators are well-known. For $a \in \mathbb{D}$, define the operator R_a by

$$R_a := \frac{1}{1 - az} \otimes \frac{1}{1 - \bar{a}z}.$$

It is easily seen that R_a is the Hankel operator $H_{\frac{1}{1-a\bar{z}}}$ and that, in fact, every rank-one Hankel operator is a scalar multiple of some R_a. Note that $\int \frac{u}{1-a\bar{z}} = u(a)$ for every $u \in \mathbf{H}^1$. Thus $R_a \in \mathcal{C}_u$ if and only if $u(a) = 0$. In particular, $\mathcal{C}_u$ (and hence $\mathcal{M}_u$) has no rank-one members when the Blaschke factor of u is trivial.

Lemma 5.5. *Suppose β is a Möbius function, while ω is outer. Then $\mathcal{C}_{\beta\omega}$ is reflexive and $\beta(S^*)A = A\beta(S)$ for each $A \in \operatorname{ref} \mathcal{C}_{\beta\omega}$.*

Proof. Choose a in the unit disc so that $\beta(z) = \frac{z-a}{1-\overline{a}z}$. Given $A \in \operatorname{ref} \mathcal{C}_{\beta\omega}$, Proposition 5.2 tells us that $\beta(S^*)A$ and $A\beta(S)$ both belong to $\operatorname{ref} \mathcal{C}_\omega = \mathcal{C}_\omega \subset \mathcal{H}ank$. Thus $\beta(S^*)(\beta(S^*)A - A\beta(S)) = 0$ (by Proposition 2.3) and the range of $\beta(S^*)A - A\beta(S)$ is contained in the one-dimensional kernel of $\beta(S^*)$. Since $\mathcal{C}_\omega$ has no rank-one members, we conclude $\beta(S^*)A - A\beta(S) = 0$ and the second conclusion of the Lemma is established.

The remaining conclusion comes from the computation

$$\begin{aligned}(S^* - aI)A &= (I - \overline{a}S^*)\beta(S^*)A \\ &= (I - \overline{a}S^*)A\beta(S) \\ &= A\beta(S)(1 - \overline{a}S) \\ &= A(S - aI),\end{aligned}$$

where the third equality follows from $A\beta(S)$ being a Hankel operator. This equality implies $S^*A = AS$, which puts $A \in \mathcal{H}ank$. □

The next proof does not directly deal with reflexivity.

Lemma 5.6. *Suppose $A \in \operatorname{ref} \mathcal{C}_{\sigma\omega}$ where σ is a singular inner function, while ω is outer. Then $\sigma(S^*)A = A\sigma(S)$.*

Proof. Recall that one can define arbitrary powers of singular inner functions. For each positive integer n and integer k, $0 \leq k \leq n$, Proposition 5.2 yields

$$\sigma^{\frac{k}{n}}(S^*)A\sigma^{\frac{n-k}{n}}(S) \in \operatorname{ref} \mathcal{C}_\omega = \mathcal{C}_\omega.$$

In particular, these operators are all in $\mathcal{H}ank$ and hence intertwine $\sigma^{\frac{1}{n}}(S^*)$ and $\sigma^{\frac{1}{n}}(S)$ (by Proposition 2.3): $\sigma^{\frac{k+1}{n}}(S^*)A\sigma^{\frac{n-k}{n}}(S) = \sigma^{\frac{k}{n}}(S^*)A\sigma^{\frac{n+1-k}{n}}(S)$. Applying these equations in turn with $k = 1, 2, \ldots, n-1$, we conclude

$$\sigma^{\frac{1}{n}}(S^*)A\sigma(S) = \sigma(S^*)A\sigma^{\frac{1}{n}}(S)$$

whence taking the limit as $n \to \infty$ yields the desired result. □

Lemma 5.7. *Suppose $A \in \operatorname{ref} \mathcal{C}_{\theta\omega}$ where θ is an inner function, while ω is outer. Then $\theta(S^*)A = A\theta(S)$.*

Proof. Set $\mathcal{F} := \{\theta$ inner: the conclusion of the Lemma holds for each outer $\omega\}$.

The preceding two lemmas imply that $\mathcal{F}$ contains all Möbius transformations and singular inner functions. We next check that $\mathcal{F}$ is closed under multiplication. Let $\theta, \eta \in \mathcal{F}$, with ω is outer, and $A \in \operatorname{ref} \mathcal{C}_{\theta\eta\omega}$. Then $\theta(S^*)A \in \operatorname{ref} \mathcal{C}_{\eta\omega}$, so

$$\eta(S^*)\theta(S^*)A = \theta(S^*)A\eta(S),$$

since $\eta \in \mathcal{F}$. But $A\eta(S) \in \operatorname{ref} \mathcal{C}_{\theta\omega}$, so

$$\theta(S^*)A\eta(S) = A\eta(S)\theta(S),$$

since $\theta \in \mathcal{F}$. Combining these equations gives $(\theta\eta)(S^*)A = A(\theta\eta)(S)$, as desired.

Suppose now that $\beta = \prod \beta_n$ is a Blaschke product, that ω is outer, and that $A \in \operatorname{ref} \mathcal{C}_{\beta\omega}$. Write $\rho_n := \prod_{k=1}^n \beta_k$ and $\tau_n := \frac{\beta}{\rho_n}$. For each n, we have $\tau_n(S^*)A \in \mathcal{C}_{\rho_n\omega}$, so

$$\rho_n(S^*)\tau_n(S^*)A = \tau_n(S^*)A\rho_n(S)$$

since $\rho_n \in \mathcal{F}$. Taking the limit on n, we see that $\beta \in \mathcal{F}$.

The result for general θ now follows from the canonical factorization of inner functions. □

Main results.

Proposition 5.8. *Let* $\mathbf{W}$ *be a weak* closed subspace of* $\mathcal{H}ank$ *with seed* θ*. Given* $B \in \operatorname{ref} \mathbf{W}$*, there is an individual Hankel operator* H_ϕ *such that* $v(S^*)Bu(S) = B\theta(S) = H_\phi$ *whenever* u, v *are inner functions satisfying* $uv = \theta$*.*

Proof. Suppose $\theta = uv$ is any factorization of θ with u and v inner functions. Fix an element of $\mathcal{C}^{-1}(\mathbf{W})$ with inner-outer factorization $\psi\omega$, so that $B \in \mathcal{C}_{\psi\omega}$ and set $\eta := \frac{\psi}{\theta}$. Then $Bu(S) \in \operatorname{ref} \mathcal{C}_{v\eta\omega}$ by Proposition 5.2. Lemma 5.7 tells us that

$$\eta(S^*)v(S^*)\left(Bu(S)\right) = \left(Bu(S)\right)v(S)\eta(S) = B\theta(S)\eta(S),$$

and the latter operator belongs to $\operatorname{ref} \mathcal{C}_\omega \subset \mathcal{H}ank$. Define

$$\mathcal{I} := \{\eta \in \mathbf{H}^\infty : \eta(S^*)v(S^*)Bu(S) = B\theta(S)\eta(S) \in \mathcal{H}ank\}$$

which is a (weak*) closed ideal. By definition of seed,

$$\gcd\left\{\frac{\psi}{\theta} : \psi \text{ inner part of functions in } \mathcal{C}^{-1}(\mathbf{W})\right\} = 1,$$

whence we conclude by Beurling's Theorem that $v(S^*)Bu(S) = B\theta(S)$ and that this operator in is $\mathcal{H}ank$. The proof is completed by taking $H_\phi = B\theta(S)$. □

Proposition 5.9. *Let* $\mathbf{W}$ *be a weak* closed subspace of* $\mathcal{H}ank$ *with seed* θ*. Then the following are equivalent for an operator* B *on* $\mathbf{H}^2$*.*

(1) $B \in \operatorname{ref} \mathcal{M}_\theta$*.*
(2) $B \in \operatorname{ref} \mathbf{W}$ *and* $B\theta(S) = 0$*.*
(3) B *satisfies* $v(S^*)Bu(S) = 0$ *whenever* u, v *are inner functions with* $uv = \theta$*.*

Proof. Assuming (1), we get $B \in \operatorname{ref} \mathbf{W}$ because $\mathcal{M}_\theta \subset \mathbf{W}$. Also, Proposition 5.2 tells us that $B\theta(S) \in \mathcal{M}_1 = \{0\}$ yielding (2).

That (2) implies (3) follows from the preceding proposition.

Finally assume (3) and suppose $f \otimes g \in (\mathcal{M}_\theta)_\perp \cap \mathbf{F}_1$. To complete the proof, we must show $\langle Bf, g\rangle = 0$. Corollary 3.8 tells us $\theta \mid fg^*$.

Take u to be the greatest common divisor of θ and the inner factor of f, and set $v := \frac{\theta}{u}$. Then v must divide g^*. Thus

$$\langle Bf, g\rangle = \left\langle B\left(u\frac{f}{u}\right), v^*\frac{g}{v^*}\right\rangle = \left\langle v(S^*)Bu(S)\left(\frac{f}{u}\right), \frac{g}{v^*}\right\rangle = 0.$$

Thus $B \perp f \otimes g$ and the proof is complete. □

Theorem 5.10. *Let* $\mathbf{W}$ *be a weak* closed subspace of* $\mathcal{H}ank$ *with seed* θ*. Then*

$$\operatorname{ref} \mathbf{W} = \mathbf{W} + \operatorname{ref} \mathcal{M}_\theta.$$

Proof. Clearly, the right-hand side is contained in the left-hand side. For the reverse inclusion, suppose $A \in \operatorname{ref} \mathbf{W}$, choose H_ϕ as in Proposition 5.8 and set $B := A - H_{\phi\bar\theta}$. Then B satisfies the third condition of Proposition 5.9 so $B \in \operatorname{ref} \mathcal{M}_\theta \subset \operatorname{ref} \mathbf{W}$. In particular, $H_{\phi\bar\theta} = A - B \in \operatorname{ref} \mathbf{W}$ and since $\mathcal{H}ank \cap \operatorname{ref} \mathbf{W} = \mathbf{W}$, we have that $H_{\phi\bar\theta} \in \mathbf{W}$. Thus $A = H_{\phi\bar\theta} + B \in \mathbf{W} + \operatorname{ref} \mathcal{M}_\theta$. □

Proof of Theorem 1.2: Observe that this is just the previous theorem. □

Corollary 5.11. *Let $N \subset \mathbf{H}^1$, take $\mathbf{W} := \mathcal{C}(N)$ to be the corresponding subspace of $\mathcal{H}ank$, and set θ as the greatest common divisor of the inner factors of the members of N. Then the following are equivalent:*

(1) $\mathbf{W}$ *is reflexive.*
(2) $\mathcal{C}_\theta$ *is reflexive.*
(3) $\mathcal{M}_\theta$ *is reflexive.*

In particular, reflexivity of $\mathbf{W}$ is equivalent to reflexivity of the largest $\mathbf{H}^\infty$-module it contains.

Proof. Proposition 3.6 implies that $\mathcal{C}^{-1}(\mathbf{W})$ is the closed linear span of N, whence θ is in fact the seed of $\mathbf{W}$. Theorem 5.10 tells us that ref $\mathcal{M}_\theta \subset \mathcal{H}ank$ if and only if ref $\mathbf{W} \subset \mathcal{H}ank$, so the equivalence of (1) and (3) follows by Corollary 3.3. Applying this general result to the special case $N = \{\theta\}$ then yields (2) $\iff$ (3). The "in particular" assertion then follows from Corollary 3.8. □

Observe that the missing converse of Corollary 5.3 (3) follows immediately from the above corollary.

Proof of Theorem 1.3: The last corollary establishes (1) $\iff$ (2) $\iff$ (3); That (3) $\iff$ (4) is the content of Theorem 4.5, while (4) $\iff$ (5) follows from Kapustin's Theorem 4.7. □

6. Finite-rank members of reflexive closures

Recall that given a singular inner function σ there is an associated finite positive Borel measure μ on $(-\pi, \pi]$ which is singular with respect to Lebesgue measure and such that, except for a constant factor of modulus 1,

$$\sigma(z) = \exp\left(-\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{it}+z}{e^{it}-z} d\mu(t)\right). \tag{2}$$

The function σ is said to be *totally atomic* if the associated measure is supported on a countable set, and *totally nonatomic* if that measure has no atoms. Observe that any singular inner function σ can then be written as $\sigma = \gamma\eta$, where γ is totally atomic and η is totally nonatomic.

The following summarizes the main results of this section.

Corollary 6.1. *Let θ be a nonconstant inner function.*

(1) *If θ is a totally nonatomic singular inner function, then* ref $\mathcal{M}_\theta$ *has no nonzero finite-rank members.*
(2) *In all other cases,* ref $\mathcal{M}_\theta$ *has rank-one members.*
(3) *In particular, when the singular factor of θ is totally atomic, then* ref $\mathcal{M}_\theta$ *is the weak* closed linear span of its rank-one members.*

The negative result (1) reflects the depth of Kapustin's work [16, 17]. In connection with (3), the relevant rank-one operators will be explicitly exhibited. As an application, we will compute ref $\mathbf{W}$ when the the singular factor of the seed of $\mathbf{W}$ is totally atomic, and when the seed is a Blaschke product.

The following proposition does not extend to operators of rank larger than one. Indeed, *each* hyperspace $\mathcal{C}_u$ has members of every rank greater than one, while $\mathcal{M}_\psi$ reduces to $\{0\}$ for outer ψ,

Proposition 6.2. *Let* $\mathbf{W}$ *be a subspace of* $\mathcal{H}ank$ *and write* θ *for its seed.*

(1) $\mathbf{W}$ *and* $\mathcal{M}_\theta$ *have the same rank-one members.*
(2) $\operatorname{ref}\mathbf{W}$ *and* $\operatorname{ref}\mathcal{M}_\theta$ *also have the same rank-one members.*

Proof. Suppose $B \in \mathbf{F}_1 \cap \operatorname{ref}\mathbf{W}$. Let $u \in \mathcal{C}^{-1}(\mathbf{W})$ have inner-outer factorization $u = \phi\omega$ so that $B \in \mathcal{C}_u$. Then $B\phi(S) \in \operatorname{ref}\mathcal{C}_\omega = \mathcal{C}_\omega$. But $\mathcal{C}_\omega$ has no rank-one members so $B\phi(S) = 0$. Define

$$\mathcal{I} := \{\psi \in \mathbf{H}^\infty : B\psi(S) = 0\},$$

which is a (weak*) closed ideal. Since θ is the greatest common divisor of the functions in $\mathcal{I}$ another application of Beurling's Theorem allows us to conclude that $B\theta(S) = 0$, whence $B \in \operatorname{ref}\mathcal{M}_\theta$ by Proposition 5.9. Thus we have shown $\mathbf{F}_1 \cap \operatorname{ref}\mathbf{W} \subset \mathbf{F}_1 \cap \operatorname{ref}\mathcal{M}_\theta$. But $\mathcal{M}_\theta \subset \mathbf{W}$, so $\mathbf{F}_1 \cap \operatorname{ref}\mathbf{W} = \mathbf{F}_1 \cap \operatorname{ref}\mathcal{M}_\theta$, proving (2). Intersecting both sides of the last equality with $\mathcal{H}ank$ gives (1) by Corollary 3.3(1). □

Pairwise divisors. The following concept relates our question to arithmetic in $\mathbf{H}^\infty$. Thanks to Jeremy Praissman for the nomenclature.

Definition 6.3. Suppose f, g, and θ are nonconstant inner functions. Then we say that (f, g) is a *pairwise divisor for* θ if whenever $\theta = uv$ is a factorization involving inner functions, then either f divides u or g divides v.

Proposition 6.4. *Suppose* (f, g) *is a pairwise divisor of* θ *while* B *is an operator on* $\mathbf{H}^2$ *satisfying* $f(S^*)B = Bg(S) = 0$. *Then* B *belongs to the (weak*) closed linear span of the rank-one members of* $\operatorname{ref}\mathcal{M}_\theta$.

Proof. Suppose $\theta = uv = vu$ is a factorization involving inner functions. If f divides v the hypothesis gives $f(S^*)B = 0$ and hence $v(S^*)B = 0$. Otherwise g divides u and hence $Bu(S) = 0$. In either case, $v(S^*)Bu(S) = 0$, so $B \in \operatorname{ref}\mathcal{M}_\theta$ by Proposition 5.9.

Now write $X := (f(S^*))^* f(S^*)$ and notice that X is an orthogonal projection. Observe that $\operatorname{kernel} f(S^*) = (f^*\mathbf{H}^2)^\perp$ is nontrivial since f is nonconstant. Fix an orthonormal basis $\{x_n\}$ of $\operatorname{kernel} f(S^*)$. Then $I - X$ is projection onto $\operatorname{kernel} f(S^*)$ and thus the series $\sum_n x_n \otimes x_n$ converges weak* to $I - X$. By hypothesis, $XB = 0$, whence

$$B = (I - X)B = \sum_n (x_n \otimes x_n)B.$$

For each n, $f(S^*)(x_n \otimes x_n)B = 0$ since $x_n \in \operatorname{kernel} f(S^*)$. Also, since $Bg(S) = 0$ we have that $(x_n \otimes x_n)Bg(S) = 0$ by orthogonality of the sequence $\{x_n\}$. Therefore, by the preceeding paragraph, each term in the above display belongs to $\operatorname{ref}\mathcal{M}_\theta$ and hence B is in the (weak*) closed linear span of the operators $(x_n \otimes x_n)B \in \operatorname{ref}\mathcal{M}_\theta$. □

Proposition 6.4 has a partial converse.

Proposition 6.5. *Suppose* $B \in \operatorname{ref}\mathcal{M}_\theta$ *has finite rank and its rank is minimal among all nonzero members of* $\operatorname{ref}\mathcal{M}_\theta$. *Then* B *has rank one and there is a pairwise divisor* (f, g) *of* θ *satisfying* $f(S^*)B = Bg(S) = 0$.

Proof. Take $\mathcal{J} := \{h \in \mathbf{H}^\infty : Bh(S) = 0\}$. Then $\mathcal{J}$ is a (weak*) closed ideal in $\mathbf{H}^\infty$, whence Beurling's Theorem provides a nonconstant inner function g such that $\mathcal{J} = g\mathbf{H}^\infty$. Thus the inner members of $\mathcal{J}$ coincide with the inner multiples of g.

Similarly, there is a nonconstant inner function f such that, for inner h, $h(S^*)B = 0$ if and only if f divides h. In particular, $f(S^*)B = Bg(S) = 0$. Now suppose $\theta = uv$ is any inner factorization of θ. By Proposition 5.9, $u(S^*)Bv(S) = 0$. Now if $Bv(S) = 0$, then g divides v. Otherwise, the rank minimality hypothesis implies $\operatorname{range} B = \operatorname{range} Bv(S)$ which implies $\operatorname{range} u(S^*)B = \operatorname{range} u(S^*)Bv(S) = \{0\}$, whence $u(S^*)B = 0$ and f divides u. This completes the proof that (f, g) is a pairwise divisor for θ. Proposition 6.4 then expresses B as a linear combination of rank-one members of $\operatorname{ref} \mathcal{M}_\theta$, whence the minimality assumption implies B itself has rank one. □

Corollary 6.6. *The following are equivalent for inner θ and nonzero $x, y \in \mathbf{H}^2$.*

(1) $x \otimes y^* \in \operatorname{ref} \mathcal{M}_\theta$.
(2) $f(S^*)x = g(S^*)y = 0$ *for some pairwise divisor (f, g) of θ.*

Proof. Assuming (1), choose (f, g) from Proposition 6.5. Then $0 = f(S^*)(x \otimes y^*) = (f(S^*)x) \otimes y^*$ implies $f(S^*)x = 0$. Also $0 = (x \otimes y^*)(g(S)) = x \otimes ((g(S))^*(y^*))$. This means $0 = (g(S))^*(y^*)$ which in turn equals $(g(S^*)(y))^*$ and we have established (2).

Conversely if (2) holds, then $0 = f(S^*)(x \otimes y^*) = (x \otimes y^*)g(S)$, and (1) follows by Proposition 6.4. □

Example 6.7. Take $\theta(z) = z^n$, and suppose i, j are nonnegative integers with $i + j \leq n - 1$. Then $z^i \otimes z^j \in \operatorname{ref} \mathcal{M}_\theta$.

Proof. (z^{i+1}, z^{j+1}) is a pairwise divisor for θ. Apply the Corollary 6.6 with $x = z^i$ and $y = z^j$. □

Application to $\operatorname{ref} \mathcal{M}_\theta$**.** It is possible to characterize all pairwise divisors.

Proposition 6.8. *Let θ be a nonconstant inner function. Then up to scalar multiples, all pairwise divisors of θ take one of the following two forms.*

(1) (β^i, β^j) *where β is is a Möbius transformation, i and j are positive integers, and β^{i+j-1} divides θ.*
(2) (α^r, α^{1-r}) *where α is a singular inner function whose measure is supported at a single point, $0 < r < 1$, and α divides θ.*

Proof. That the displayed pairs work follows from uniqueness of the canonical Blaschke-singular factorization of inner functions.

For the converse, let (f, g) be a pairwise divisor for θ. The factorization $\theta = \theta \cdot 1 = 1 \cdot \theta$ forces f and g to divide θ. Suppose first that $f(a) = 0$ for some $a \in \mathbb{D}$ and write β for the corresponding Möbius transformation. There is a highest power of β which divides θ which thus factors as $\theta = \rho\beta^n$ for some ρ which is relatively prime to β. Since f does not divide ρ, we conclude that $g = \beta^j$ for some $1 \leq j \leq n$. But we can also factor $\theta = \beta^{n-j+1}[\beta^{j-1}\rho]$. Since g doesn't divide the bracketed factor, we conclude that that $f = \beta^i$ for some $i \leq n - j + 1$ so our pairwise divisor takes the form (1).

It remains to consider the case where f and g are singular inner functions. Since any Blaschke factor of θ is irrelevant to questions of divisibility by f and g, we may as well assume that θ is singular inner as well. Write μ for the measure on the unit circle associated to θ as in Equation (2). Then inner divisors of θ correspond to measures which are less than or equal to μ setwise. For inner divisors h and k of θ, we write $h \ll k$ and $h \perp k$ when the corresponding measures are related in these ways. Applying the Jordan Decomposition Theorem, we can factor $\theta = \rho\sigma$ where $\rho \perp f$ and $\sigma \ll f$. Since f does not divide ρ, we conclude g must divide σ so in fact $g \ll \sigma \ll f$. By symmetry f and g must be mutually absolutely continuous. But now, if h is any nontrivial divisor of f, then (h, g) is still a pairwise divisor of θ whence $h \approx g \approx f$. In other words, the measure associated to f is mutually absolutely continuous with every smaller measure, and this forces the measures of f and g to be supported at a single common point of $\mathbb{T}$. Take $\alpha := fg$ to see that (f, g) takes the form (2). □

Corollary 6.9. *Let σ be a totally nonatomic singular inner function. Then σ does not have a pairwise divisor.*

Proof. This is an immediate consequence of Proposition 6.8. □

Proposition 6.10. *Suppose θ is a power of a Möbius function. Then* ref $\mathcal{M}_\theta$ *is the (weak*) closed span of its rank-one members.*

Proof. Express $\theta = \beta^n$ where β is the Möbius function $\beta(z) = \frac{z-a}{1-\bar{a}z}$ and write V for the isometry $\beta(S)$. Let $B \in \operatorname{ref} \mathcal{M}_\theta$. Since $BV^n = 0$, we have

$$B = \sum_{j=0}^{n-1} BV^j (I - VV^*)(V^*)^j.$$

Fix j and write $C := BV^j(I - VV^*)(V^*)^j$. By Proposition 5.9 we have that $\beta^{n-j}(S^*)C = 0$. Also, since $(I - VV^*)V = 0$ we have $C\beta^{j+1}(S) = 0$. Since $(\beta^{n-j}, \beta^{j+1})$ is a pairwise divisor for β^n, by Proposition 6.4, we conclude that each term in the preceding display is in the (weak*) closed linear span of the rank-one members of ref $\mathcal{M}_\theta$ and the proof is complete. □

Proposition 6.11. *Suppose θ is a singular inner function whose measure is supported at a single point. Then* ref $\mathcal{M}_\theta$ *is the (weak*) closed linear span of its rank-one members.*

Proof. We must show $\mathbf{F}_1 \cap \operatorname{ref} \mathcal{M}_\theta$ is total in ref $\mathcal{M}_\theta$. Let $B \in \operatorname{ref} \mathcal{M}_\theta$.

Let V be the isometry $\theta(S)$. Observe that for each natural number n, since $BV = 0$, we have

$$BV^{\frac{1}{n}} = \sum_{k=1}^{n-1} BV^{\frac{1}{n}} V^{\frac{k-1}{n}} \left(I - V^{\frac{1}{n}} (V^*)^{\frac{1}{n}} \right) (V^*)^{\frac{k-1}{n}}$$

Fix k and write $C := BV^{\frac{k}{n}} \left(I - V^{\frac{1}{n}} (V^*)^{\frac{1}{n}} \right) (V^*)^{\frac{k-1}{n}}$. By Proposition 5.9, we have $\theta^{1-\frac{k}{n}}(S^*)C = 0$. Also, since $(I - V^{\frac{1}{n}}(V^*)^{\frac{1}{n}})V^{\frac{1}{n}} = 0$, it follows that $CV^{\frac{k}{n}} = 0$. Since $(\theta^{1-\frac{k}{n}}, \theta^{\frac{k}{n}})$ is a pairwise divisor for θ, Proposition 6.4 tells us that C belongs to the (weak*) closed span of $\mathbf{F}_1 \cap \operatorname{ref} \mathcal{M}_\theta$. Thus each term in the preceeding display belongs to the (weak*) closed linear span of $\mathbf{F}_1 \cap \operatorname{ref} \mathcal{M}_\theta$ and hence $BV^{\frac{1}{n}}$ belongs to the (weak*) closed linear span of $\mathbf{F}_1 \cap \operatorname{ref} \mathcal{M}_\theta$. Taking the limit as $n \to \infty$ yields $B \in [\mathbf{F}_1 \cap \operatorname{ref} \mathcal{M}_\theta]$ as desired. □

All the pieces are now in place for the proof of Corollary 6.1.

Theorem 6.12. *Let θ be an inner function having factorization $\theta = \gamma\sigma$, where γ is an inner function with singular factor totally atomic and σ is a totally nonatomic singular inner function. Then*

$$[\mathbf{F}_1 \cap \operatorname{ref} \mathcal{M}_\theta] = [\mathbf{F} \cap \operatorname{ref} \mathcal{M}_\theta] = \operatorname{ref} \mathcal{M}_\gamma.$$

Proof. The inclusion $[\mathbf{F}_1 \cap \operatorname{ref} \mathcal{M}_\theta] \subset [\mathbf{F} \cap \operatorname{ref} \mathcal{M}_\theta]$ is obvious.

Suppose $B \in \mathbf{F} \cap \operatorname{ref} \mathcal{M}_\theta$. Then $B\gamma(S) \in \mathbf{F} \cap \operatorname{ref} \mathcal{M}_\sigma$. But Corollary 6.9 tells us σ has no pairwise divisors, whence Proposition 6.5 implies $\operatorname{ref} \mathcal{M}_\sigma$ has no nonzero members of finite rank. Thus $B\gamma(S) = 0$. On the other hand, $B\sigma(S) \in \operatorname{ref} \mathcal{M}_\gamma$. Since γ and σ are relatively prime, there exists $\mathbf{H}^\infty$ functions h and k such that $1 = \gamma h + \sigma k$ and hence $B = B\gamma(S)h(S) + B\sigma(S)k(S) = B\sigma(S)k(S)$. By Proposition 5.2(3), we can then conclude that $B \in \operatorname{ref} \mathcal{M}_\gamma$. Thus we have established the second inclusion $[\mathbf{F} \cap \operatorname{ref} \mathcal{M}_\theta] \subset \operatorname{ref} \mathcal{M}_\gamma$.

For the remaining inclusion, express $\gamma = \Pi_n \gamma_n$ where each γ_n is either a power of a Möbius function or a singular inner function whose measure is supported at a single point; no two γ_n's should have a common zero or be supported at the same point of $\mathbb{T}$. Applying Propositions 6.10 and 6.11, we learn that for each n,

$$\operatorname{ref} \mathcal{M}_{\gamma_n} = [\mathbf{F}_1 \cap \operatorname{ref} \mathcal{M}_{\gamma_n}] \subset [\mathbf{F}_1 \cap \operatorname{ref} \mathcal{M}_\theta].$$

But Proposition 5.3(1) tells us that $\operatorname{ref} \mathcal{M}_\gamma$ is the closed linear span of $\{\operatorname{ref} \mathcal{M}_{\gamma_n}\}$. Thus $\operatorname{ref} \mathcal{M}_\gamma \subset [\mathbf{F}_1 \cap \operatorname{ref} \mathcal{M}_\theta]$ and the cycle is complete. □

Proof of Corollary 6.1: Observe that, for θ inner, $\mathcal{M}_\theta \neq \{0\}$ if and only if θ is nonconstant. Then, we only need to apply the previous theorem with $\gamma = 1$, γ nonconstant, and $\sigma = 1$ to get Parts (1), (2), and (3) respectively. □

We close this subsection with a refinement of Corollary 6.1(1).

Proposition 6.13. *Let $\mathbf{W}$ be a weak*-closed subspace of $\mathcal{H}ank$ whose seed θ is a totally nonatomic singular inner function. Then all finite-rank members of* ref $\mathbf{W}$ *belong to* $\mathbf{W}$ *itself.*

Proof. Let $A \in \operatorname{ref} \mathbf{W}$ have finite rank. Apply Proposition 5.8 to find an individual Hankel operator H_ϕ such that

$$v(S^*)Au(S) = A\theta(S) = H_\phi$$

whenever u, v are inner functions satisfying $uv = \theta$. Since H_ϕ has finite rank, the vectors $H_\phi 1, H_\phi z, H_\phi z^2, \dots$ are linearly dependent and there is a polynomial $p \not\equiv 0$ such that $0 = H_\phi p = H_{p\phi} 1$. Since 1 is a separating vector for $\mathcal{H}ank$, it follows that $H_{p\phi} = 0$ and thus $A\theta(S)p(S) = 0$. Multiplying the last display through by $p(S)$ yields

$$v(S^*)Ap(S)u(S) = Ap(S)\theta(S) = 0$$

whenever $uv = \theta$. But now Proposition 5.9 yields $Ap(S) \in \operatorname{ref} \mathcal{M}_\theta$. In view of Corollary 6.1(1), this means $Ap(S) = 0$, and we have already observed that $A\theta(S) \in \mathcal{H}ank$. Since p and θ are relatively prime, there exist $\mathbf{H}^\infty$ functions h and k such that $1 = ph + \theta k$. Hence $A = Ap(S)h(S) + A\theta(S)k(S) = A\theta(S)k(S) \in \mathcal{H}ank$ and we get $A \in \mathcal{H}ank \cap \operatorname{ref} \mathbf{W} = \mathbf{W}$ as desired. □

Computing reflexive closures. Theorem 5.10 can be used to compute reflexive closures of nonreflexive spaces. For example, (matrices of) operators in $\mathbf{W} := \mathcal{M}_{z^k}$ are supported on the top k skew diagonals and hence the same is true for members of ref $\mathbf{W}$. On the other hand, Example 6.7 shows that all matrices supported on this triangle do in fact belong to ref $\mathbf{W}$. Since there are precisely k independent Hankel operators supported on that triangle, the codimension of $\mathbf{W}$ in ref $\mathbf{W}$ (which quantifies how nonreflexive $\mathbf{W}$ is) is $\frac{k(k-1)}{2}$. Theorem 5.10 now tells us how to compute the reflexive closures of all Hankel spaces having seed z^k. The following result generalizes this analysis.

Corollary 6.14. *Suppose θ is an inner function whose singular factor is totally atomic and set*

$$\Re := \{x \otimes y^* : f(S^*)x = g(S^*)y = 0 \textit{ for some pairwise divisor } (f,g) \textit{ of } \theta\}.$$

(1) ref $\mathcal{M}_\theta$ *is the weak* closed linear span of* $\Re$.
(2) *If* $\mathbf{W}$ *is any weak* closed subspace of* $\mathcal{H}ank$ *with seed* θ*, then* ref $\mathbf{W} = \mathbf{W} + \overline{\operatorname{span} \Re}$.

Proof. For (1), combine Corollaries 6.1(3) and 6.6.
For (2), apply Theorem 5.10 to the result of Part (1). □

Of course, the description of $\Re$ can hardly be considered concrete, though Proposition 6.8 helps. The set $\mathcal{C}$ given in Corollary 6.15 is is more satisfying in this respect.

Corollary 6.15. *Suppose θ is a Blaschke product and set*

$$\mathcal{C} := \left\{ \frac{z^{i-1}}{(1-az)^i} \otimes \frac{z^{j-1}}{(1-\overline{a}z)^j} \ : i,j \in \mathbb{N}, \left(\frac{z-a}{1-\overline{a}z}\right)^{i+j-1} \textit{ divides } \theta \right\}.$$

(1) ref $\mathcal{M}_\theta$ *is the (weak*) closed linear span of* $\mathcal{C}$.
(2) *If* $\mathbf{W}$ *is any (weak*) closed subspace of* $\mathcal{H}ank$ *with seed* θ*, then* ref $\mathbf{W} = \mathbf{W} + \overline{\operatorname{span} \mathcal{C}}$.

Proof. In this case, all pairwise divisors of θ take the form of Proposition 6.8(1). Thus the set $\Re$ of the last corollary becomes

$$\Re := \{x \otimes y^* : \beta^i(S^*)x = \beta^j(S^*)y = 0 \text{ for some Möbius } \beta \text{ with } \beta^{i+j-1} \text{ dividing } \theta\}.$$

But observe that, if β is the Möbius function $\beta(z) = \frac{z-a}{1-\overline{a}z}$, then

$$\operatorname{kernel}(\beta^n(S^*)) = \left((\beta^*)^n \mathbf{H}^2\right)^\perp = \operatorname{span}\left\{ \frac{z^k}{(1-az)^{k+1}} \ : k = 0, \ldots, n-1 \right\},$$

and thus the span of $\Re$ equals the span of $\mathcal{C}$. □

Corollary 6.16. *If θ is an inner function having a multiple root or a nontrivial atomic inner factor, then* ref $\mathcal{M}_\theta$ *has a non-Hankel member of rank one.*

Proof. Since $\phi|\theta$ implies ref $\mathcal{M}_\phi \subset$ ref $\mathcal{M}_\theta$, it suffices to consider the cases when
(1) $\theta = \beta^2$ for β a Möbius transformation vanishing at $a \in \mathbb{D}$ and
(2) $\theta = \alpha$ where α is a singular inner function whose measure is supported at a single point in $\mathbf{T}$.

In Case (1) Corollary 6.15 tells us $\frac{1}{1-az} \otimes \frac{z}{(1-\overline{a}z)^2} \in \operatorname{ref} \mathcal{M}_\theta$ and we know this operator does not belong to $\mathcal{H}ank$ because it is not a scalar multiple of R_a.

In Case (2), Corollary 6.1(2) tells us ref $\mathcal{M}_\theta$ has a rank-one element which can't be Hankel since $\mathcal{M}_\theta$ itself has no rank-one elements (because θ has no roots). □

Relation to earlier papers. A main goal of our work to this point has been to identify precisely which (weak* closed) subspaces of $\mathcal{H}ank$ are reflexive. The following summarizes the results we have obtained.

Theorem 6.17. *Let* $\mathbf{W}$ *be a weak* closed subspace of* $\mathcal{H}ank$*, and write* θ *for its seed, i.e.,* $\mathcal{M}_\theta$ *is the largest submodule of* $\mathcal{H}ank$ *contained in* $\mathbf{W}$*. Consider the canonical factorization* $\theta = \beta\alpha\sigma$*, where* β *is a Blaschke product,* α *is a totally atomic singular inner function and* σ *is a totally nonatomic singular inner function. Let* μ *the singular measure corresponding to the singular inner function* σ*. Then,*

1. W *is reflexive if and only if* $\mathcal{M}_\theta$ *is reflexive if and only if* $\mathcal{C}_\theta$ *is reflexive if and only if* $S(\theta)$ *is reflexive.*
2. $\mathcal{M}_\theta$ *is reflexive if and only if* $\mathcal{M}_\beta$*,* $\mathcal{M}_\alpha$ *and* $\mathcal{M}_\sigma$ *are reflexive.*
3. $\mathcal{M}_\beta$ *is reflexive if and only if* β *has no repeated roots.*
4. $\mathcal{M}_\alpha$ *is reflexive if and only if* $\alpha = 1$*.*
5. $\mathcal{M}_\sigma$ *is reflexive if and only if* μ *is a BC-vanishing measure.*

Earlier proofs of the Jordan model analogues of Parts (3) and (4) can be found in [29]. (5) depends strongly on Kapustin's deep work, and is unlikely to be simplified. On the other hand, the proofs we have given of everything except (5) are independent of existing papers on Jordan models:

- (1) follows from Corollary 5.11 and Theorem 4.5,
- (2) is Corollary 5.3(2),
- the 'if' implication of (3) comes from Lemma 5.5 and Corollary 5.3(2),
- the 'if' implication of (4) comes from Corollary 5.1,
- the 'only if' implications of (3) and (4) result from Corollary 6.16.

In particular, we were able to use operators of rank one to get the negative content of (3) and (4). Proposition 6.13 explains why such an approach cannot help with (5).

7. Semi–infinite Dimensional

Let us turn our attention to the so-called semi–infinite dimensional case. First, for each $n \in \mathbb{N}$, let $\mathcal{P}_n$ denote the set of all polynomials with degree less than or equal to n, thought of as a subspace of $\mathbf{H}^2$. The space $\mathbf{B}(\mathcal{P}_n, \mathbf{H}^2)$ is dual to the trace class $\mathbf{Tr}(\mathbf{H}^2, \mathcal{P}_n)$ via the bilinear form

$$\langle A, T\rangle = \operatorname{trace}(AT), \qquad A \in \mathbf{B}(\mathcal{P}_n, \mathbf{H}^2), \quad T \in \mathbf{Tr}(\mathbf{H}^2, \mathcal{P}_n).$$

For $n \in \mathbb{N}$ and $\phi \in \overline{\mathbf{H}^2}$ define $H_\phi : \mathcal{P}_n \longrightarrow \mathbf{H}^2$ by $H_\phi f = PJ(\phi f)$, where P denotes the orthogonal projection onto $\mathbf{H}^2$ and J is the flip operator $J(f)(z) := f(\overline{z})$. Thus, H_ϕ is an operator in $\mathbf{B}(\mathcal{P}_n, \mathbf{H}^2)$. It is easy to check that the matrix of H_ϕ with respect to the canonical bases $\{z^k\}_{k=0}^n$ and $\{z^k\}_{k=0}^\infty$, for $\mathcal{P}_n$ and $\mathbf{H}^2$ respectively, has constant skew–diagonals. Conversely, it is easy to see that all Hankel operators are of the form H_ϕ, as above.

Since $\|H_\phi z^j\| \le \|\phi\|$ for $j = 0 \dots n$ with equality for $j = 0$, it follows that

$$\|H_\phi\| \le \sqrt{n+1}\, \|\phi\| = \sqrt{n+1}\, \|H_\phi(1)\|, \quad \phi \in \overline{\mathbf{H}^2}. \tag{3}$$

We denote the set of all Hankel operators in $\mathbf{B}(\mathcal{P}_n, \mathbf{H}^2)$ as $\mathcal{H}ank(\mathcal{P}_n, \mathbf{H}^2)$.

For each $H_\phi \in \mathcal{H}ank(\mathcal{P}_n, \mathbf{H}^2)$, $f \in \mathcal{P}_n$ and $g \in \mathbf{H}^2$, a straightforward calculation shows that

$$\langle H_\phi, f \otimes g \rangle = \langle H_\phi f, g \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} \phi(e^{i\theta}) f(e^{i\theta}) g^*(e^{i\theta}) d\theta.$$

Every element T of $\mathbf{Tr}(\mathbf{H}^2, \mathcal{P}_n)$ has a unique expression $T = \sum_{j=0}^n z^j \otimes g_j(z)$, and we define its *cosymbol* $\Gamma(T) := \sum_{j=0}^n z^j g_j^*(z)$. The preceding display then generalizes to

$$\langle H_\phi, T \rangle = \frac{1}{2\pi} \int \phi \Gamma(T), \quad \phi \in \overline{\mathbf{H}^2}, \quad T \in \mathbf{Tr}(\mathbf{H}^2, \mathcal{P}_n). \tag{4}$$

Proposition 7.1. (1) *For each $g \in \mathbf{H}^2$, we have $\Gamma(1 \otimes g^*) = g$.*
(2) *Γ is bounded and has range $\mathbf{H}^2$.*
(3) *The kernel of Γ is the preannihilator of $\mathcal{H}ank(\mathcal{P}_n, \mathbf{H}^2)$*

Proof. (1) is a matter of definition and (2) is easy to check. (3) follows from Equation (4). □

Theorem 7.2. *Let $n \in \mathbb{N}$. The space $\mathcal{H}ank(\mathcal{P}_n, \mathbf{H}^2)$ is transitive and elementary.*

Proof. Suppose $f \otimes g \in \mathcal{H}ank(\mathcal{P}_n, \mathbf{H}^2)_\perp$. Then Equation (4) yields $\int \phi f g^* = 0$ for all $\phi \in \overline{\mathbf{H}^2}$. But this means $fg^* \equiv 0$ whence $f = 0$ or $g = 0$. Thus $\mathcal{H}ank(\mathcal{P}_n, \mathbf{H}^2)_\perp$ has no rank-one members and $\mathcal{H}ank(\mathcal{P}_n, \mathbf{H}^2)$ is transitive.

Given $T \in \mathbf{Tr}(\mathbf{H}^2, \mathcal{P}_n)$, we can write $T = 1 \otimes (\Gamma(T))^* + [T - 1 \otimes (\Gamma(T))^*]$. Since the bracketed expression is in the kernel of Γ, we have $\mathbf{Tr}(\mathbf{H}^2, \mathcal{P}_n) = \mathbf{F}_1 + \mathcal{H}ank(\mathcal{P}_n, \mathbf{H}^2)_\perp$ and $\mathcal{H}ank(\mathcal{P}_n, \mathbf{H}^2)$ is elementary. □

In particular, an appropriate version of Proposition 2.5 (in our semi-infinite dimensional setting) tells us no proper subspace of $\mathcal{H}ank(\mathcal{P}_n, \mathbf{H}^2)$ is transitive.

We now study subspaces of $\mathcal{H}ank(\mathcal{P}_n, \mathbf{H}^2)$. Because $\mathcal{P}_n$ is finite-dimensional, every norm closed linear manifold in $\mathbf{B}(\mathcal{P}_n, \mathbf{H}^2)$ is automatically weak* closed, so we need not worry about the latter topology. Equation (3) shows that the symbol map is is a Banach space isomorphism between $\overline{\mathbf{H}^2}$ and $\mathcal{H}ank(\mathcal{P}_n, \mathbf{H}^2)$. Moreover, taking perps relative to the bilinear form

$$\langle f, g \rangle = \frac{1}{2\pi} \int fg, \quad f \in \overline{\mathbf{H}^2}, g \in \mathbf{H}^2$$

sets up a one-to-one correspondence between subspaces of $\overline{\mathbf{H}^2}$ and subspaces of $\mathbf{H}^2$. Thus the companion spaces of the following definition are all subspaces of $\mathcal{H}ank(\mathcal{P}_n, \mathbf{H}^2)$, and every subspace of $\mathcal{H}ank(\mathcal{P}_n, \mathbf{H}^2)$ is accounted for in this way.

Definition 7.3. The *companion space* to a sub*set* N of $\mathbf{H}^2$ is defined as

$$\mathcal{C}^n(N) := \{H_\phi \in \mathcal{H}ank(\mathcal{P}_n, \mathbf{H}^2) : \int \phi u = 0 \text{ for each } u \in N\}.$$

For an individual $u \in \mathbf{H}^2$, we write $\mathcal{C}^n_u := \mathcal{C}^n(\{u\})$.

In particular, $\mathcal{C}^n_u$ is the most general hyperplane in $\mathcal{H}ank(\mathcal{P}_n, \mathbf{H}^2)$.

Lemma 7.4. *Let $n \in \mathbb{N}$, $f \in \mathcal{P}_n$, $g \in \mathbf{H}^2$ and $N \subset \mathbf{H}^2$.*

(1) *In order that a trace class operator T belong to $(\mathcal{C}^n(N))_\perp$ it is necessary and sufficient that $\Gamma(T)$ belong to the closed linear span $[N]$ of N.*

(2) *In particular,* $f \otimes g \in (\mathcal{C}^n(N))_\perp$ *if and only if* fg^* *belongs to* $[N]$.

Proof. For necessity in (1), assume $\Gamma(T) \in [N]$ and let $H_\phi \in \mathcal{C}^n(N)$. By definition, the last statement implies $\int \phi u = 0$ for all $u \in N$. Thus $\int \phi\Gamma(T) = 0$ as well, whence $\langle H_\phi, T\rangle = 0$ by Equation 4. Thus $T \in (\mathcal{C}^n(N))_\perp$ as required.

For the converse, suppose $\Gamma(T) \notin [N]$ and choose $\phi \in \overline{\mathbf{H}^2}$ with $\int \phi u = 0$ for all $u \in N$ but $\int \phi\Gamma(T) \neq 0$. The last two conditions respectively mean $H_\phi \in \mathcal{C}^n(N)$, but $\langle H_\phi, T\rangle \neq 0$ and we have shown that $T \notin (\mathcal{C}^n(N))_\perp$.

(2) is a special case of (1) since $\Gamma(f \otimes g) = fg^*$. □

Corollary 7.5. *Let* $n \in \mathbb{N}$, $f \in \mathcal{P}_n$, $g \in \mathbf{H}^2$ *and* $u \in \mathbf{H}^2$. *Then* $f \otimes g \in (\mathcal{C}^n_u)_\perp$ *if and only if there exists* $\lambda \in \mathbb{C}$ *with* $fg^* = \lambda u$.

Note also that $\mathbf{B}(\mathcal{P}_0, \mathbf{H}^2)$ is reflexive and elementary (thus every subspace is reflexive [1, Proposition 2.10]). The main result of this section is the analog of Theorem 1.3 in the semi–infinite dimensional setting.

Theorem 7.6. *Suppose* $n \in \mathbb{N}$ *and* $u \in \mathbf{H}^2$. *Then* $\mathcal{C}^n_u$ *is reflexive if and only if the inner factor of* u *is a Blaschke factor with no repeated roots.*

The following lemma depends on the fact that 1 is a strictly separating vector for $\mathcal{H}ank(\mathcal{P}_n, \mathbf{H}^2)$ (Equation (3)). Whence this argument cannot be used in the infinite dimensional setting or the finite dimensional setting!

Lemma 7.7. *Let* $\mathbf{M}$ *be a subspace of* $\mathcal{H}ank(\mathcal{P}_n, \mathbf{H}^2)$. *The following statements are equivalent:*

(1) *The space* $\mathbf{M}$ *is reflexive,*
(2) *If* $A1 = 0$, *and* $A \in \operatorname{ref}\mathbf{M}$, *then* $A = 0$.

Proof. Assume that (1) holds, $A1 = 0$, and $A \in \operatorname{ref}\mathbf{M}$. Since $A \in \mathbf{M}$ we have that A is Hankel, whence $A = 0$ since 1 is a separating vector.

Now assume that (2) holds and that $A \in \operatorname{ref}\mathbf{M}$. We must show that $A \in \mathbf{M}$. Since $A \in \operatorname{ref}\mathbf{M}$ we have that $A1 \in \overline{\mathbf{M}(1)}$, by definition of reflexivity. This amounts to saying that there exist a sequence of operators $\{A_n\}$ in $\mathbf{M}$ with $\|A_n1 - A1\| \to 0$. This implies that the sequence $\{A_n1\}$ is Cauchy, and since 1 is a strictly separating vector, that $\{A_n\}$ is also a Cauchy sequence. Since $A_n \in \mathbf{M}$ and $\mathbf{M}$ is closed, it follows that there exists an operator $A_0 \in \mathbf{M} \subset \operatorname{ref}\mathbf{M}$ with $A_n \to A_0$. Hence, we have that $(A - A_0)1 = 0$ and $A - A_0 \in \operatorname{ref}\mathbf{M}$. The hypothesis implies that $A - A_0 = 0$. Therefore, $A = A_0 \in \mathbf{M}$. □

Our technique for showing hyperplanes of Hankel operators to be nonreflexive is to exhibit non-Hankel operators in their reflexive closure. We should remind the reader that we use P_n to denote the operator $P_n : \mathbf{H}^2 \longrightarrow \mathcal{P}_n$, the orthogonal projection onto the polynomials of degree at most n.

Proposition 7.8. *Let* $n \in \mathbb{N}$ *and fix* $a \in \mathbb{D}$. *Then the non-Hankel rank-one operator* $\frac{z}{(1-az)^2} \otimes P_n \frac{1}{1-\overline{a}z}$ *belongs to the reflexive closure of* $\mathcal{C}^n((z-a)^2\mathbf{H}^2)$.

Proof. Suppose $f \in \mathcal{P}_n$ and $g \in \mathbf{H}^2$ with $f \otimes g \in (\mathcal{C}^n((z-a)^2\mathbf{H}^2))_\perp$. Then, by Lemma 7.4, fg^* has a repeated root at a. Notice that

$$\begin{aligned}\left\langle \frac{z}{(1-az)^2} \otimes P_n \frac{1}{1-\overline{a}z}, f \otimes g \right\rangle &= \left\langle f, P_n \frac{1}{1-\overline{a}z} \right\rangle \left\langle \frac{z}{(1-az)^2}, g \right\rangle \\ &= \left\langle f, \frac{1}{1-\overline{a}z} \right\rangle \left\langle g^*, \frac{z}{(1-\overline{a}z)^2} \right\rangle\end{aligned}$$

and since $\frac{1}{1-\bar{a}z}$ acts as a reproducing kernel in $\mathcal{P}_n$ while $\frac{z}{(1-\bar{a}z)^2}$ is a reproducing kernel for the first derivative in $\mathbf{H}^2$, we then have that

$$\left\langle \frac{1}{1-az} \otimes P_n \frac{z}{(1-\bar{a}z)^2}, f \otimes g \right\rangle = f(a) g^{*\prime}(a).$$

But if fg^* has a repeated root at a then either f or $g^{*\prime}$ has a root at a, which means the previous equation equals zero. □

Proposition 7.9. *Let $n > 0$. Suppose σ is a nontrivial singular inner function. Choose r to be a nonzero member of $(\sigma^*\mathbf{H}^2)^\perp$ (necessarily nonconstant). Then the non-Hankel operator $r \otimes 1$ belongs to the reflexive closure of $\mathcal{C}^n(\sigma\mathbf{H}^2)$.*

Proof. Suppose $f \in \mathcal{P}_n$ and $g \in \mathbf{H}^2$ with $f \otimes g \in (\mathcal{C}^n(\sigma\mathbf{H}^2))_\perp$. Then σ divides fg^* by Lemma 7.4. Since $f \in \mathcal{P}_n$, the singular inner function σ must divide g^*. Since σ^* divides g, it follows that $g \in \sigma^*\mathbf{H}^2$ and thus $\langle r, g\rangle = 0$. Notice then that $\langle r \otimes 1, f \otimes g\rangle = \langle f, 1\rangle \langle r, g\rangle = 0$. That $r \otimes 1$ is not Hankel is clear since r is not constant (look at its matrix). □

The following proposition isolates the function theory needed in applying Lemma 7.7 in the proof of Theorem 7.11. If $a \in \mathbb{D} \setminus \{0\}$ let $b_a(z) = \frac{|a|}{a} \frac{a-z}{1-\bar{a}z}$, and if $a = 0$ let $b_a(z) = z$.

Proposition 7.10. *Let $\mathcal{F} \subset \mathbf{H}^2$ and consider the following subspace of $\mathbf{H}^2$*

$$N := \left[\left\{ \frac{u(z)}{1-bz} \right\}_{b \in \mathbb{D}}, \left\{ \frac{u(z)}{z-c} \right\}_{u(c)=0} \; : \; u \in \mathcal{F} \right].$$

Suppose $\beta := \gcd\{$inner factor of $u : u \in \mathcal{F}\}$ is a Blaschke product with no repeated roots. Then $N = \mathbf{H}^2$.

Proof. First observe that $\mathcal{F} \subset N$. For each $u \in \mathcal{F}$ we have that

$$\left[\left\{ \frac{u(z)}{1-bz} \right\}_{b \in \mathbb{D}} \right] = \left[\{z^n u(z)\}_{n \in \mathbb{N}} \right]$$

and

$$z \, \frac{u(z)}{z-c} = u(z) + c \, \frac{u(z)}{z-c} \text{ whenever } u(c) = 0.$$

Hence, $N \subset \mathbf{H}^2$ is invariant under multiplication by z and by Beurling's Theorem $N = \phi\mathbf{H}^2$ for some inner function ϕ. Since ϕ must divide each $u \in \mathcal{F}$, we see it must divide β, and since ϕ divides $\frac{u(z)}{z-c}$ for each root c of β, we conclude that ϕ has no roots at all and hence must be constant. □

Theorem 7.11. *Let K be a subset of $\mathbf{H}^2$, and take γ to be the greatest common divisor of the inner factors of members of K. Then the following are equivalent.*

(1) *The companion space $\mathbf{M}$ of K is reflexive.*
(2) *γ is a Blaschke product without repeated roots.*

Proof. Suppose first that γ has a repeated root a. Then $K \subset (z-a)^2\mathbf{H}^2$ so $\mathcal{C}^n((z-a)^2\mathbf{H}^2) \subset \mathbf{M}$. By Proposition 7.8, we know $\mathcal{C}^n((z-a)^2\mathbf{H}^2)$ is nonreflexive, so the larger space $\mathbf{M}$ cannot be reflexive either. Similarly, if γ has a non trivial singular inner factor σ, then $\mathbf{M}$ contains the nonreflexive space $\mathcal{C}^n(\sigma\mathbf{H}^2)$ and hence is nonreflexive as well.

For the converse, assume γ is a Blaschke product with no repeated roots and let $A \in \operatorname{ref} \mathbf{M}$. Assume that $A1 = Az = \cdots = Az^k = 0$ for some fixed k, $0 \le k < n$. We will show that $Az^{k+1} = 0$. Then by induction we will get that, if $A1 = 0$, then $A = 0$. Whence $\mathbf{M}$ will be reflexive by Lemma 7.7.

Given $a, b \in \mathbb{D}$, and $u \in K$, we have by Lemma 7.4 that

$$(1 - az)^k (1 - bz) \otimes \frac{u^*(z)}{(1 - \overline{a}z)^k (1 - \overline{b}z)} \in \mathbf{M}_\perp$$

and, since $A \in \operatorname{ref} \mathbf{M}$, it follows that

$$A(1 - az)^k (1 - bz) \;\perp\; \frac{u^*(z)}{(1 - \overline{a}z)^k (1 - \overline{b}z)}.$$

The induction hypothesis then implies, for nonzero $a, b \in \mathbb{D}$, that

$$Az^{k+1} \;\perp\; \frac{u^*(z)}{(1 - \overline{a}z)^k (1 - \overline{b}z)}.$$

But since $\left\| \frac{1}{(1-\overline{a}z)^k} - 1 \right\|_\infty \to 0$ as $a \to 0$, we have that

$$\left\| \frac{u^*(z)}{(1 - \overline{a}z)^k (1 - \overline{b}z)} - \frac{u^*(z)}{(1 - \overline{b}z)} \right\|_2 \to 0, \qquad \text{as} \quad a \to 0.$$

Hence,

$$Az^{k+1} \;\perp\; \frac{u^*(z)}{1 - \overline{b}z}$$

for all $b \in \mathbb{D}$ (the case $b = 0$ obtained by taking the limit as $b \to 0$.)

Using similar reasoning, but starting with the fact that, for $c \in \mathbb{D}$ with $u(c) = 0$, the operator

$$(1 - az)^k (z - c) \otimes \frac{u^*(z)}{(1 - \overline{a}z)^k (z - \overline{c})} \in \mathbf{M}_\perp$$

for all $a \in \mathbb{D}$, we obtain that $Az^{k+1} \;\perp\; \dfrac{u^*(z)}{z - \overline{c}}$ whenever $u(c) = 0$. Thus the vector Az^{k+1} is perpendicular to the space N, as defined in Proposition 7.10. But $N = \mathbf{H}^2$ by that Proposition and thus $Az^{k+1} = 0$.

By induction $A = 0$ and therefore $\mathbf{M}$ is reflexive. □

Proof of Theorem 7.6: Set $K := \{u\}$ in Theorem 7.11. □

Proof of Theorem 1.4: The equivalency of conditions (2) and (3) is just Theorem 7.6. Now, choose $K \subset \mathbf{H}^2$ such that $\mathbf{M} = \mathcal{C}^n(K)$ in Theorem 7.11 to obtain the equivalency of (1) and (3). □

8. Finite Dimensional

We now consider the finite dimensional setting. For each $n \in \mathbb{N}$ we let $\mathcal{P}_n := \{p \in \mathbb{C}[z] : \deg p \le n\}$. Given $m, n \in \mathbb{N}$ and $\phi \in \operatorname{span}\{1, \overline{z}, \ldots, \overline{z}^{m+n}\}$ define $H_\phi : \mathcal{P}_n \to \mathcal{P}_m$ as

$$H_\phi f = P_m J(\phi f),$$

where P_m denotes the orthogonal projection onto $\mathcal{P}_m$, and J is the flip operator $J(p)(z) := p(\overline{z})$. Thus, H_ϕ is an operator in $\mathbf{B}(\mathcal{P}_n, \mathcal{P}_m)$ whose matrix relative to the standard (polynomial) bases has constant skew–diagonals, and clearly all Hankel operators arise in this manner. The full space of Hankel operators in $\mathbf{B}(\mathcal{P}_n, \mathcal{P}_m)$ will be denoted by $\mathcal{H}ank(\mathcal{P}_n, \mathcal{P}_m)$. In this case there is a one-to-one relation between a Hankel operator and its symbol.

Notice also that for each $H_\phi \in \mathcal{H}ank(\mathcal{P}_n, \mathcal{P}_m)$, $f \in \mathcal{P}_n$ and $g \in \mathcal{P}_m$, we have that

$$\langle H_\phi f, g\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} \phi(e^{i\theta}) f(e^{i\theta}) g^*(e^{i\theta}) dt, \tag{5}$$

Again, $\langle H_\phi f, g\rangle$ only depends on the product fg^* and not on the individual factors f and g^*.

The following theorem is proved for the $n \times n$ case in [1, Example 3.4]. Using the exact same argument (or adapting arguments from the previous sections), one can show the following.

Theorem 8.1. *Let $n, m \in \mathbb{N}$.*

1. *$\mathcal{H}ank(\mathcal{P}_n, \mathcal{P}_m)$ has dimension $n + m + 1$.*
2. *$\mathcal{H}ank(\mathcal{P}_n, \mathcal{P}_m)$ is transitive.*
3. *$\mathcal{H}ank(\mathcal{P}_n, \mathcal{P}_m)$ is elementary.*

In [1, Proposition 4.7], it is shown that if a subspace $\mathbf{M}$ of $\mathbf{B}(\mathcal{P}_n, \mathcal{P}_m)$ is transitive, then $\dim \mathbf{M} \geq n + m + 1$. Therefore, no proper subspace of $\mathcal{H}ank(\mathcal{P}_n, \mathcal{P}_m)$ can be transitive; the same conclusion follows from a slight modification of Part 2 of Corollary 3.3.

Definition 8.2. Let $n, m \in \mathbb{N}$. For a polynomial $u \in \mathcal{P}_{m+n}$ we define $\mathcal{C}_u^{m,n}$ in $\mathcal{H}ank(\mathcal{P}_n, \mathcal{P}_m)$ as

$$\mathcal{C}_u^{m,n} := \left\{ H_\phi \in \mathcal{H}ank(\mathcal{P}_n, \mathcal{P}_m) : \frac{1}{2\pi}\int_{-\pi}^{\pi} \phi(e^{i\theta}) u(e^{i\theta}) d\theta = 0 \right\}.$$

Since, $\dim \mathcal{H}ank(\mathcal{P}_n, \mathcal{P}_m) = \dim \mathcal{P}_{m+n}$, this is the most general hyperplane in $\mathcal{H}ank(\mathcal{P}_n, \mathcal{P}_m)$.

Lemma 8.3. *Let $m, n \in \mathbb{N}$, $f \in \mathcal{P}_n$ and $g \in \mathcal{P}_m$. Then, $f \otimes g \in (\mathcal{C}_u^{m,n})_\perp$ if and only if $fg^* = \lambda u$ for some $\lambda \in \mathbb{C}$*

Proof. Assume $fg^* = \lambda u$ for some $\lambda \in \mathbb{C}$ and let $H_\phi \in \mathcal{C}_u^{m,n}$. By definition, the last statement implies $\int \phi u = 0$. Thus $\int \phi f g^* = 0$ as well, whence $\langle H_\phi f, g\rangle = 0$ by Equation (5). Thus $f \otimes g \in (\mathcal{C}_u^{m,n})_\perp$ as required.

Conversely, assume $fg^* \notin [u]$. As in Lemma 7.4, choose $\phi \in \operatorname{span}\{1, \overline{z}, \dots, \overline{z}^{m+n}\}$ such that $\int \phi u = 0$ but $\int \phi f g^* \neq 0$. It then follows that while $H_\phi \in \mathcal{C}_u^{m,n}$ also $\langle H_\phi, f \otimes g\rangle \neq 0$, and hence $f \otimes g \notin (\mathcal{C}_u^{m,n})_\perp$. ☐

Proposition 8.4. *Fix $m, n \in \mathbb{N}$ and let $u \in \mathcal{P}_{m+n}$. Then*

1. *if $\deg u < m + n - 1$, then $\mathcal{C}_u^{m,n}$ is not reflexive,*
2. *if u has a repeated root, then $\mathcal{C}_u^{m,n}$ is not reflexive.*

Proof. Let $f \in \mathcal{P}_n$ and $g \in \mathcal{P}_m$ with $f \otimes g \in (\mathcal{C}_u^{m,n})_\perp$ and thus $fg^* = \lambda u$.

Let us show (1). Observe that the rank-one operator $z^{m-1} \otimes z^n$ is not a Hankel operator. However, if the degree of the polynomial u is strictly less than $m + n - 1$ and $fg^* = \lambda u$, then either

(a) the degree of f equals n, which implies that the degree g is less than or equal to $m - 2$,

(b) or the degree of f is strictly less than n.

In either case, we have that $f \otimes g$ is orthogonal to $z^{m-1} \otimes z^n$. That is, the reflexive closure of $\mathcal{C}_u^{m,n}$ contains a non-Hankel operator.

To show (2), suppose that u has a repeated root a, that is $(z-a)^2 | u$ which implies that either $(z-a)^2 \mid f$ or $(z-a) \mid g^*$. Let us define

$$x := \sum_{k=0}^{m} a^k z^k \quad \text{and} \quad y := \sum_{k=0}^{n-1} (k+1)\overline{a}^k z^{k+1}.$$

Direct computation confirms that x acts as a reproducing kernel for evaluation at $\overline{a}$ while y acts as a reproducing kernel for derivative evaluation at a:

$$\langle u, x\rangle = u(\overline{a}), \quad u \in \mathcal{P}_m; \qquad \langle v, y\rangle = v'(a), \quad v \in \mathcal{P}_n.$$

We will show that $f \otimes g$ is orthogonal to the non-Hankel rank-one operator $x \otimes y$ (to see that $x \otimes y$ is a non-Hankel operator just observe that $\langle (x \otimes y)1, z\rangle = 0$ but $\langle (x \otimes y)z, 1\rangle = 1$ and thus its matrix is not constant on the skew-diagonals).

Thus we have $\langle x \otimes y, f \otimes g\rangle = \langle x, g\rangle \langle f, y\rangle = g^*(a) f'(a) = 0$. Hence the non-Hankel operator $x \otimes y$ is in the reflexive closure of $\mathcal{C}_u^{m,n}$ and we have that $\mathcal{C}_u^{m,n}$ is not reflexive. □

The converse of Proposition 8.4 is the main result of this section.

Theorem 8.5. *Fix $m, n \in \mathbb{N}$ and let $u \in \mathcal{P}_{m+n}$. If the polynomial u has no repeated roots and $\deg u \geq m+n-1$, then $\mathcal{C}_u^{m,n}$ is reflexive.*

The operators in the next definition can be thought of as "analytic Toeplitz operators acting between different spaces". Given $\phi \in \mathcal{P}_1$ we let

$$T_\phi : \mathcal{P}_{n-1} \longrightarrow \mathcal{P}_n,$$

be the linear operator defined by $T_\phi(f) = \phi f$.

Proposition 8.6. *Let $B \in \mathbf{B}(\mathcal{P}_n, \mathcal{P}_m)$ where $n \geq 2$. The following are equivalent:*

(1) $B \in \mathcal{H}ank(\mathcal{P}_n, \mathcal{P}_m)$,
(2) $BT_\phi \in \mathcal{H}ank(\mathcal{P}_{n-1}, \mathcal{P}_m)$ *for all* $\phi \in \mathcal{P}_1$,
(3) $BT_1, BT_z \in \mathcal{H}ank(\mathcal{P}_{n-1}, \mathcal{P}_m)$.

Proof. (1) implies (2) implies (3) is clear. It remains to prove that (3) implies (1). It suffices to show that each entry of the matrix associated to the operator B, with respect to standard orthonormal bases for $\mathcal{P}_m$ and $\mathcal{P}_n$, is a Hankel matrix. For $1 \leq k \leq n-1$ and $1 \leq l \leq m$ we have that

$$\begin{aligned} \left\langle Bz^k, z^{l-1}\right\rangle &= \left\langle (BT_1)z^k, z^{l-1}\right\rangle \\ &= \left\langle (BT_1)z^{k-1}, z^l\right\rangle \\ &= \left\langle Bz^{k-1}, z^l\right\rangle. \end{aligned}$$

Now, for $k = n$ and $1 \leq l \leq m$ we have that

$$\begin{aligned} \left\langle Bz^k, z^{l-1}\right\rangle &= \left\langle (BT_z)z^{k-1}, z^{l-1}\right\rangle \\ &= \left\langle (BT_z)z^{k-2}, z^l\right\rangle \\ &= \left\langle Bz^{k-1}, z^l\right\rangle. \end{aligned}$$

Therefore the associated matrix of B is a Hankel matrix which is equivalent to $B \in \mathcal{H}ank(\mathcal{P}_n, \mathcal{P}_m)$. □

Proposition 8.7. *Let $u \in \mathcal{P}_{m+n-1}$ and $\phi \in \mathcal{P}_1$. Then $\mathcal{C}_{u\phi}^{m,n} T_\phi \subset \mathcal{C}_u^{m,n-1}$.*

Proof. Let $H_\psi \in \mathcal{C}_{u\phi}^{m,n}$. Then $\int \psi u \phi = 0$. Since $H_{\psi\phi} = H_\psi T_\phi$ and $\int (\psi\phi) u = 0$ we have that $H_\psi T_\phi \in \mathcal{C}_u^{m,n-1}$ as desired. □

Before proving Theorem 8.5, we show a special case of it.

Proposition 8.8. *$\mathcal{C}_u^{1,1}$ is reflexive if u has no repeated roots and* $\deg u \geq 1$.

Proof. $\mathcal{C}_u^{1,1}$ is 2-dimensional, so $\dim((\mathcal{C}_u^{1,1})_\perp) = 4 - 2$ is 2, as well. Hence showing $\mathcal{C}_u^{1,1}$ is reflexive amounts to finding two linearly independent rank-one members of $(\mathcal{C}_u^{1,1})_\perp$.

First suppose that $\deg u = 2$, that is $u(z) = (z-a)(z-b)$ for distinct values $a, b \in \mathbb{C}$. Then by Lemma 8.3 we have that

$$(z-a) \otimes (z-\overline{b}) \qquad (z-b) \otimes (z-\overline{a})$$

are linearly independent rank-one members of $(\mathcal{C}_u^{1,1})_\perp$.

Now suppose that $\deg u = 1$, that is $u(z) = (z-a)$ for some $a \in \mathbb{C}$. Then we have that, as before,

$$(z-a) \otimes 1 \qquad 1 \otimes (z-\overline{a})$$

are linearly independent rank-one members of $(\mathcal{C}_u^{1,1})_\perp$. □

Proof of Theorem 8.5: Consider the following statement:

$$P(m,n) : \text{if } u \in \mathcal{P}_{m+n} \text{ has no repeated roots and } \deg u \geq m+n-1,$$

$$\text{then } \mathcal{C}_u^{m,n} \text{ is reflexive.}$$

Let $m, n \in \mathbb{N}$ and assume that $P(m,n)$ is true. Now let $u \in \mathcal{P}_{m+n+1}$ be such that $\deg u \geq m+n$ and u has no repeated roots, let $B \in \operatorname{ref} \mathcal{C}_u^{m,n+1}$ and let $a \neq b$ be two distinct roots of u. Then

$$\begin{aligned} BT_{z-a} \in \left(\operatorname{ref} \mathcal{C}_u^{m,n+1}\right) T_{z-a} \quad &\subset \quad \operatorname{ref} \left(\mathcal{C}_u^{m,n+1} T_{z-a}\right) \\ &= \quad \mathcal{C}_{\frac{u}{z-a}}^{m,n} \\ &\subset \quad \mathcal{H}ank(\mathcal{P}_n, \mathcal{P}_m), \end{aligned}$$

where we are using the fact that if $\mathbf{M}$ is a subspace of operators and T an operator, then $(\operatorname{ref} \mathbf{M})\, T \subset \operatorname{ref} (\mathbf{M}\, T)$, (which can be found in [2] Lemma 4.5), combined with Proposition 8.7 and the reflexivity of $\mathcal{C}_{\frac{u}{z-a}}^{m,n}$.

Similarly, $BT_{z-b} \in \mathcal{H}ank(\mathcal{P}_n, \mathcal{P}_m)$. Since $1, z \in \operatorname{span}\{z-a, z-b\}$ we have that $BT_1, BT_z \in \mathcal{H}ank(\mathcal{P}_n, \mathcal{P}_m)$. Hence by Proposition 8.6, the operator $B \in \mathcal{H}ank(\mathcal{P}_{n+1}, \mathcal{P}_m)$. Thus $\mathcal{C}_u^{m,n+1}$ is reflexive and $P(m, n+1)$ is true. So we have that

$$\forall\, m, n \in \mathbb{N} \quad [P(m,n) \implies P(m,n+1)]. \tag{6}$$

By Proposition 8.8 we have that $P(1,1)$ is true and (6) implies that for all $n \in \mathbb{N}$ that $P(1,n)$ is true. Now observe that $(\mathcal{C}_u^{m,n})^* = \mathcal{C}_{u^*}^{n,m}$. Thus by symmetry we have that for all $m \in \mathbb{N}$ that $P(m,1)$ is true. Applying (6) again we get that for all $m, n \in \mathbb{N}$ that $P(m,n)$ is true. □

We will need the following classical result.

Theorem 8.9 (Bertini's Theorem). *Suppose $p(z)$ and $q(z)$ are polynomials (over $\mathbb{C}$) with no common roots. Then $p(z)+tq(z)$ has a repeated root for at most finitely many $t \in \mathbb{C}$.*

Proof. If a is a repeated root of the polynomial $p(z) + tq(z)$ for some $t \in \mathbb{C}$, then $p(a)q'(a) - p'(a)q(a) = 0$. Thus there are at most $\deg(p) + \deg(q) - 1$ numbers a satisfying this equation and thus only a finite number of a's can be repeated roots for $p(z) + tq(z)$, regardless of t.

If the same a is a repeated root for the polynomials $p(z)+tq(z)$ and $p(z)+sq(z)$ for some $s, t \in \mathbb{C}$ such that $s \neq t$, then, p and q share a common root, which contradicts the hypothesis. Hence, once an element of the finite set described above is a repeated root of $p(z) + tq(z)$, it cannot be a repeated root of any other $p(z) + sq(z)$, $t \neq s$. Hence, the polynomial $p(z) + tq(z)$ has a repeated root for at most finitely many $t \in \mathbb{C}$. □

Proposition 8.10. *Let $F \subset \mathbb{C}$ be a finite nonempty set and suppose $k > 1$. If $w_1, \ldots, w_k$ are relatively prime polynomials, i. e. $\gcd(w_1, \ldots, w_k) = 1$, then there exist $t_2, \ldots, t_k \in \mathbb{C}$ such that the polynomial*

$$w_1 + t_2 w_2 + \cdots + t_k w_k$$

has no repeated roots, no roots in F, and degree $\geq \deg(w_1)$.

Proof. Let $F \subset \mathbb{C}$ be a finite nonempty set. Let w_1, w_2 be polynomials which are relatively prime.

If for some $a \in F$ we can choose $s \neq t \in \mathbb{C}$ such that a is a root of both the polynomials $w_1 + sw_2$ and $w_1 + tw_2$ then a would be a common root of w_1 and w_2, contradicting our assumption that w_1 and w_2 are relatively prime polynomials. It follows from the finiteness of F that there can only be finitely many t for which $w_1 + tw_2$ has a root in F. In view of Bertini's Theorem, for all but finitely many $t \in \mathbb{C}$, the polynomial $w_1 + tw_2$ has no repeated roots as well as no roots in F. Also, there can be at most one t for which $\deg(w_1 + tw_2) < \deg(w_1)$.

Now assume $k > 2$ and that the Proposition holds for relatively prime $(k-1)$-tuples of polynomials. Suppose we are given relatively prime polynomials $w_1, ..., w_k$. Set $g := \gcd(w_1, \ldots, w_{k-1})$, and take F' the set of roots of w_k. Applying (part of) the inductive hypothesis, we obtain $t_2, ..., t_{k-1} \in \mathbb{C}$ such that the polynomial $\frac{w_1}{g} + t_2 \frac{w_2}{g} + \cdots + t_{k-1} \frac{w_{k-1}}{g}$ has no roots in F' and its degree is at least $\deg \frac{w_1}{g}$. Since g has no roots in common with w_k, we conclude that $w_1 + t_2 w_2 + \cdots + t_{k-1} w_{k-1}$ is relatively prime to w_k. An application of the first paragraph of the proof completes the inductive argument. □

In view of the analogous statement to Proposition 2.5 in our context, any reflexive subspace of a reflexive hyperspace of $\mathcal{H}ank(\mathcal{P}_n, \mathcal{P}_m)$ will also be reflexive. The following generalized version of Theorem 8.5 shows that conversely every reflexive subspace of $\mathcal{H}ank(\mathcal{P}_n, \mathcal{P}_m)$ is contained in a reflexive hyperspace.

Corollary 8.11. *Fix $m, n \in \mathbb{N}$, let $u_1, \ldots, u_k \in \mathcal{P}_{m+n}$, and set $r = \gcd(u_1, \ldots, u_k)$. Then the following are equivalent:*

(1) $\bigcap_{j=1}^{k} \mathcal{C}_{u_j}^{m,n}$ *is reflexive.*

(2) $\bigcap_{j=1}^{k} \mathcal{C}_{u_j}^{m,n}$ *is contained in a reflexive hyperspace of $\mathcal{H}ank(\mathcal{P}_n, \mathcal{P}_m)$.*

(3) *Some u_i has degree at least $m+n-1$ and $r(z)$ has no repeated roots.*

Proof. We will first show that (1) implies (3). Suppose that r has a repeated root and and let $a \in \mathbb{C}$ be such a root. Consider the polynomials $k_0, k_1 \in \mathcal{P}_{n+m}$ defined to have the property that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \overline{k_0(e^{i\theta})}p(e^{i\theta})d\theta = p(a) \quad \text{and} \quad \frac{1}{2\pi}\int_{-\pi}^{\pi} \overline{k_1(e^{i\theta})}p(e^{i\theta})d\theta = p'(a)$$

(these polynomials are easily seen to exist: see the proof of Proposition 8.4).

Since $(z-a)^2$ divides $u_1, u_2, \ldots, u_k$, we have that for all j, $H_{\overline{k_0}}$ and $H_{\overline{k_1}}$ belong to $\mathcal{C}^{m,n}_{u_j}$. Thus

$$\operatorname{span}\{H_{\overline{k_0}}, H_{\overline{k_1}}\} \subset \bigcap_{j=1}^{k} \mathcal{C}^{m,n}_{u_j},$$

whence ref (span $\{H_{\overline{k_0}}, H_{\overline{k_1}}\}$) is contained in ref $\left(\bigcap_{j=1}^{k} \mathcal{C}^{m,n}_{u_j}\right)$. But one can easily check that the non-Hankel operator $k_0 \otimes k_1 \in$ ref (span $\{H_{\overline{k_0}}, H_{\overline{k_1}}\}$). Therefore ref $\left(\bigcap_{j=1}^{k} \mathcal{C}_{u_j}\right)$ contains a non-Hankel member, so $\bigcap_{j=1}^{k} \mathcal{C}^{m,n}_{u_j}$ is not reflexive.

If every u_i has degree less than $m+n-1$, then one can check, in a similar way to the arguments above, that ref $\left(\bigcap_{j=1}^{k} \mathcal{C}_{u_j}\right)$ contains the non-Hankel operator $z^{m-1} \otimes z^n$ and $\bigcap_{j=1}^{k} \mathcal{C}^{m,n}_{u_j}$ is not reflexive.

We will now show that (3) implies (2). When $k=1$, we have $r = u_1$ and it suffices to apply Theorem 8.5. Thus assume $k > 2$ and for definiteness that u_1 has degree at least $m+n-1$.

Let $r = \gcd(u_1, ..., u_k)$ and suppose that r has no repeated roots. Next, let $v_j = u_j/r$ for $j = 1, ..., k$. Thus, by Proposition 8.10 we can choose $t_2, ..., t_k \in \mathbb{C}$ such that

$$v_1 + t_2 v_2 + \cdots + t_k v_k$$

has no repeated roots, no roots in common with r and its degree is at least that of v_1. Multiplying through by r, we see that

$$u_1 + t_2 u_2 + \cdots + t_k u_k$$

has no repeated roots and has degree at least $m+n-1$.

Thus Theorem 8.5 tells us $\mathcal{C}^{m,n}_{u_1+t_2u_2+\cdots+t_ku_k}$ is reflexive.

Finally (2) implies (1) since $\mathcal{H}ank(\mathcal{P}_n, \mathcal{P}_m)$ is elementary. □

Proof of Theorem 1.5: Observe that Corollary 8.11 is just Theorem 1.5. □

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