Equiangular Tight Frames and Fourth Root Seidel Matrices

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Abstract

In this paper we construct complex equiangular tight frames (ETFs). In particular, we study the grammian associated with an ETF whose off-diagonal entries consist entirely of fourth roots of unity. These ETFs are classified, and we also provide some computational techniques which give rise to previously undiscovered ETFs.

Keywords: Equiangular Tight Frames, Seidel Matrix, Antisymmetric Matrix
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1. Introduction

Recently several different methods for constructing equiangular tight frames (ETFs) have been explored. In [11], a partial list of pairs (n, k), which admit complex ETFs is, determined by studying analysis operators satisfying the property that each entry of the scaled matrix $\sqrt{k}V^*$ (V is an analysis operator) is a p^{th} root of unity. A correspondence between difference sets and equiangular cyclic frames is given in [8]. Bodmann, Paulsen, and Tomforde, [2], provide necessary and sufficient conditions for the existence of *Seidel matrices* with two eigenvalues whose off-diagonal entries are all cube roots of unity. Finding Seidel matrices with two eigenvalues is known to be equivalent to the existence of ETFs, see [7].

In this paper, we study the existence and construction of Seidel matrices with two eigenvalues (equivalently ETFs) whose off-diagonal entries are all fourth roots of unity. Some of our methods are similar to that of [2]. However, unlike the cube roots of unity case, we are able to show that a certain class of real skew-symmetric matrices yield complex ETFs. In addition, we provide necessary and sufficient conditions for a certain class of fourth root Seidel matrices to have exactly two eigenvalues. It is worth noting that in [2], the authors take advantage of known results about certain regular directed graphs to construct complex ETFs. Although the fourth roots of unity case can be translated into

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a problem in graph theory, we were unable to find any known results in the literature to construct complex ETFs of this type.

This paper is organized as follows. We complete the introduction by defining a *Seidel matrix* and stating a crucial result relating Seidel matrices to ETFs. Section 2 describes our extension of Seidel matrices to include fourth roots of unity and some direct consequences of this generalization. In Section 3, we use real skew-symmetric matrices to construct complex ETFs. Section 4 covers the construction of complex ETFs which do not arise from previously known ETFs or real skew-symmetric matrices.

1.1. Seidel Matrices and ETFs

Both of the papers [3] and [9] provide an excellent introduction to the general theory on frames as well as a good read. However, for a detailed discussion on equiangular frames, the motivation behind this paper, the authors recommend reading [1, 7].

The following definition and theorem are due to Holmes and Paulsen [7].

Definition 1.1. An $n \times n$ self-adjoint matrix Q such that $q_{ii} = 0$ and $|q_{ij}| = 1$ for all $i \neq j$ is called a **Seidel matrix**.

Note that some authors refer to a Seidel matrix as a *signature matrix*.

Theorem 1.2 (Theorem 3.3 of [7]). *Let Q be a self-adjoint $n \times n$ matrix with $q_{ii} = 0$ and $|q_{ij}| = 1$ for all $i \neq j$. Then the following are equivalent:*

1. *Q is the Seidel matrix of an ETF,*
2. *$Q^2 = (n-1)I + \mu Q$ for some necessarily real number μ,*
3. *Q has exactly two eigenvalues.*

The focus of this paper is to construct Seidel matrices with exactly two eigenvalues whose off diagonal entries are all fourth roots of unity. We shall see that condition (2) in Theorem 1.2 is particularly useful for the computational aspects of this construction.

A Seidel matrix Q satisfying any of the three equivalent conditions in Theorem 1.2 yields several useful parameters. It is shown in [7], if $\lambda_1 < 0 < \lambda_2$ are Q's two eigenvalues, then the parameters $n,\ k,\ \mu,\ \lambda_1$, and λ_2 satisfy the following properties:

$$\mu = (n-k)\sqrt{\frac{n-1}{k(n-k)}} = \lambda_1 + \lambda_2, \qquad k = \frac{n}{2} - \frac{\mu n}{2\sqrt{4(n-1)+\mu^2}} \tag{1}$$

$$\lambda_1 = -\sqrt{\frac{k(n-1)}{n-k}}, \qquad \lambda_2 = \sqrt{\frac{(n-1)(n-k)}{k}}, \quad n = 1 - \lambda_1\lambda_2.$$

2. Preliminaries

We begin by introducing some new definitions and preliminary results which will prove useful throughout this discussion.

Definition 2.1. A self-adjoint matrix with all diagonal entries equal to zero and all nondiagonal entries equal to complex fourth roots of unity will be called a **fourth root Seidel matrix**. A fourth root Seidel matrix S is said to be in **standard form** if all of the entries in the first row and column are one, except for s_{11}.

Note that conjugating by the appropriate diagonal matrix will transform a fourth root Seidel matrix into standard form. This process leads to an equivalence relation on fourth root Seidel matrices where the matrices in standard form are class representatives.

Theorem 1.2 connects equiangular frames to Seidel matrices with two eigenvalues. This motivates the following proposition (and subsequent corollary) which is similar to Proposition 2.4 of [2]. Notice that an $n \times n$ fourth root Seidel matrix S with two eigenvalues satisfies the equation

$$S^2 = (n-1)I + \mu S$$

for some real number μ.

Proposition 2.2. *Let S be a fourth root Seidel matrix in standard form satisfying the equation*

$$S^2 = (n-1)I + \mu S$$

and $x_j = \#\{k|S_{kj} = 1\}$. Then $e_j := \frac{n+\mu-2x_j}{2}$ is the number of entries in the j^{th} column equal to i, for $j > 1$. Furthermore, in the j^{th} column, e_j is the number of entries equal to $-i$, and the number of entries equal to -1 is $\frac{n-\mu-2e_j-2}{2}$.

Proof. For $1 < j \leq n$, define

$$\begin{aligned} y_j &:= \#\{k|S_{kj} = i\} \\ z_j &:= \#\{k|S_{kj} = -1\} \\ t_j &:= \#\{k|S_{kj} = -i\}. \end{aligned}$$

For $1 < j \leq n$,

$$\mu = \mu S_{1j} = [(n-1)I + \mu S]_{1j} = \left[S^2\right]_{1j} = (x_j - 1) + y_j i + z_j(-1) + t_j(-i)$$

which gives

$$(x_j - \mu - 1 - z_j) + (y_j - t_j)i = 0.$$

Thus, $y_j = t_j$ and $z_j = x_j - \mu - 1$. Since the j^{th} column has $n-1$ nonzero entries, we have

$$x_j + y_j + z_j + t_j = n - 1.$$

Substituting for z_j and t_j, we are left with

$$y_j = \frac{n + \mu - 2x_j}{2}.$$

Corollary 2.3. *The difference between the number of $1's$ and the number of $-1's$ in a column is $\mu + 1$. Furthermore, μ is an integer.*

Proof.

$$\begin{aligned} x_j - \frac{n - \mu - 2e_j - 2}{2} &= x_j - \frac{n - \mu - (n + \mu - 2x_j) - 2}{2} \\ &= x_j - (-\mu + x_j - 1) \\ &= \mu + 1. \end{aligned}$$

Searching for fourth root Seidel matrices using brute force is only feasible for "small" values of n. Given a particular n, the following proposition bounds the possible values of μ.

Proposition 2.4. *Let S be an $n \times n$ fourth root Seidel matrix in standard form satisfying $S^2 = (n-1)I + \mu S$. Then $n - 2 > \mu > 2 - n$.*

Proof. By Corollary 2.3, $\mu + 1$ is the number of ones minus the number of negative ones in all columns except possibly the first column. The largest this can be is $n - 1$ and the smallest is $3 - n$. So $n - 1 \geq \mu + 1 \geq 3 - n$ or $n - 2 \geq \mu \geq 2 - n$.

Note that Proposition 2.2 also implies that μ is even if and only if n is even, and Proposition 2.4 gives us a list of possible values of μ for each n. Evaluating Equation (1) at possible values of n and μ and checking to see if k is an integer, gives the possible values for n, μ and k. The values corresponding to $4 < n \leq 30$ are listed in Table 1.

<table>
<tr><th>n</th><th>μ</th><th>k</th><th>n</th><th>μ</th><th>k</th></tr>
<tr><td>4</td><td>0</td><td>2</td><td>18</td><td>0</td><td>9</td></tr>
<tr><td>6</td><td>0</td><td>3</td><td>20</td><td>0</td><td>10</td></tr>
<tr><td>8</td><td>0</td><td>4</td><td>22</td><td>0</td><td>11</td></tr>
<tr><td>10</td><td>0</td><td>5</td><td>24</td><td>0</td><td>12</td></tr>
<tr><td>12</td><td>0</td><td>6</td><td>26</td><td>0</td><td>13</td></tr>
<tr><td>14</td><td>0</td><td>7</td><td rowspan="3">28</td><td>-6</td><td>21</td></tr>
<tr><td rowspan="3">16</td><td>-2</td><td>10</td><td>0</td><td>14</td></tr>
<tr><td>0</td><td>8</td><td>6</td><td>7</td></tr>
<tr><td>2</td><td>6</td><td>30</td><td>0</td><td>15</td></tr>
</table>

Table 1: Possible $4 < n \leq 30, \mu, k$ values

3. Constructing Complex ETFs using Real Matrices

Here we present a method for constructing fourth root Seidel matrices from real skew symmetric matrices whose entries are all ± 1. Note that, if A is any such matrix, then iA is a fourth root Seidel matrix.

Proposition 3.1. *Let A be a real matrix with all diagonal entries equal to zero and all off diagonal entries equal to ± 1 such that $A^T = -A$. Then the standard form, S, of iA has entries $s_{j,k} = \pm i$ for $j > 1$, $k > 1$ and $j \neq k$.*

Proof. Let A be a real $n \times n$ matrix such that $A^T = -A$, $a_{jj} = 0$, and all off diagonal entries are equal to ± 1. Fix D as the diagonal matrix with $d_{11} = 1$ and $d_{jj} = -ia_{1j}$ for $j > 1$. Then the matrix $S = D^*(iA)D$ is in standard form. For $j > 1$, $k > 1$, and $j \neq k$,

$$s_{jk} = d_{jj}^*(ia_{jk})d_{kk} = (ia_{1j})(ia_{jk})(-ia_{1k}) = \pm i.$$

Corollary 3.2. *Let S denote the standard form of a fourth root Seidel matrix with two eigenvalues.*

1. *If $s_{j,k} = \pm 1$ for $j > 1$, $k > 1$, and $j \neq k$, then S corresponds to a real equiangular frame.*
2. *If $s_{j,k} = \pm i$ for $j > 1$, $k > 1$, and $j \neq k$, then S corresponds to a complex equiangular frame arising from a skew symmetric matrix (as described in Proposition 3.1).*
3. *If S is does not fit (1) or (2) above then S corresponds to a "truly" complex equiangular frame.*

We will refer to fourth root Seidel matrices with two eigenvalues (and the corresponding frames) mentioned above as **real** (R), **skew-symmetric** (SS), and **truly complex** (TC) respectively. We now provide some restrictions on the existence of fourth root Seidel matrices with two eigenvalues obtained from skew symmetric matrices which expedite our computer search for these frames.

Proposition 3.3. *Let A be a real $n \times n$ matrix with two eigenvalues such that $A^T = -A$, and $a_{jk} = \pm 1$ for $j \neq k$, then $-A^2 = (n-1)I$.*

Proof. Clearly, $b = 0$ for A to satisfy $-A^2 = (n-1)I + b(iA)$.

The following theorem uses the structure of the fourth root Seidel matrices.

Theorem 3.4. *Let A be a real $n \times n$ matrix with two eigenvalues such that $A^T = -A$, and $a_{jk} = \pm 1$ for $j \neq k$, then $n = 2$ or $n \equiv 0 \mod 4$.*

Proof. When $n = 2$, the matrix

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

satisfies our conditions.

Suppose $n \geq 4$. Without loss of generality, assume $a_{1j} = 1$ for $2 \leq j \leq n$ and $a_{23} = 1$. For $3 \leq j \leq n$, define

$$\begin{aligned} C_{++} &:= \#\{j | a_{2j} = 1 \text{ and } a_{3j} = 1\} \\ C_{+-} &:= \#\{j | a_{2j} = 1 \text{ and } a_{3j} = -1\} \\ C_{-+} &:= \#\{j | a_{2j} = -1 \text{ and } a_{3j} = 1\} \\ C_{--} &:= \#\{j | a_{2j} = -1 \text{ and } a_{3j} = -1\}. \end{aligned}$$

Since $a_{23} = 1$, and the rows of A are orthogonal, we get

$$C_{++} + C_{+-} - C_{-+} - C_{--} = -1 \text{ and}$$
$$C_{++} - C_{+-} + C_{-+} - C_{--} = 1$$

by considering the inner product of rows 2 and 3 with row 1. The inner product of rows 2 and 3 gives the equation

$$C_{++} - C_{+-} - C_{-+} + C_{--} = -1.$$

Lastly, the inner product of row 2 with itself gives the equation

$$C_{++} + C_{+-} + C_{-+} + C_{--} = n - 3.$$

The relation $4C_{++} = n - 4$ follows from combining the previous four equations. Since n is an integer, it must be divisible by 4.

Example 3.5. *Let*

$$A = \begin{pmatrix} 0 & 1 & -1 & 1 \\ -1 & 0 & -1 & -1 \\ 1 & 1 & 0 & -1 \\ -1 & 1 & 1 & 0 \end{pmatrix}.$$

Clearly, A satisfies $A = -A^t$, and the eigenvalues of A are $\pm i\sqrt{3}$. Thus, the matrix iA satisfies $iA = (iA)^*$ and has eigenvalues $\pm\sqrt{3}$. Interestingly, iA is a Grammian matrix for the complex equiangular $(4, 2)$ frame.

While Theorem 3.4 tells us where to look for skew-symmetric matrices with nondiagonal entries equal to ± 1, it does not guarantee the existence of any such matrices. However, example 3.5 does show that such a matrix exists when $n = 4$. The following proposition goes further to show that the existence of one such square matrix of dimension n, guarantees the existence of another with dimension $2n$.

Proposition 3.6. *If M is a matrix of dimension n such that $M^T = -M$ and $M^2 = (1 - n)I_n$, then the matrix*

$$N = \begin{pmatrix} -M & M - I_n \\ M + I_n & M \end{pmatrix}$$

satisfies $N^T = -N$ and $N^2 = (1 - 2n)I_{2n}$.

Proof.

$$\begin{aligned} N^2 &= \begin{pmatrix} 2 * M^2 - I_n & 0 \\ 0 & 2 * M^2 - I_n \end{pmatrix} \\ &= \begin{pmatrix} (1 - 2n)I_n & 0 \\ 0 & (1 - 2n)I_n \end{pmatrix} \\ &= (1 - 2n)I_{2n}. \end{aligned}$$

Applying Proposition 3.6 to the matrix in Example 3.5 yields an 8×8 antisymmetric matrix corresponding to a skew-symmetric frame. Repeatedly applying this proposition yields frames for $n = 4 \cdot 2^k$ where k is any nonnegative integer. We have also constructed antisymmetric matrices for $n = 12$ and $n = 20$ satisfying $M^2 = (1-n)I_n$ yielding two infinite families of frames for $n = 12 \cdot 2^k$ and $n = 20 \cdot 2^k$. Table 2 summarizes our results for fourth root Seidel matrices with two eigenvalues coming from real matrices.

n	μ	k	R?	SS?	n	μ	k	R?	SS?
4	0	2	N[1]	Y (Known)	18	0	9	Y[1]	N (Thm. 3.4)
6	0	3	Y[1]	N (by Const.)	20	0	10	N[1]	Y (by Const.)
8	0	4	N[1]	Y (by Const.)	22	0	11	N[1]	N (Thm. 3.4)
10	0	5	Y[1]	N (by Const.)	24	0	12	N[1]	Y (Prop. 3.6)
12	0	6	N[1]	Y (by Const.)	26	0	13	Y[1]	N (Thm. 3.4)
14	0	7	N[1]	N (Thm. 3.4)		-6	21	Y[1]	N (Prop. 3.3)
	-2	10	Y[1]	N (Prop. 3.3)	28	0	14	N[1]	Y (by Const.)
16	0	8	N[1]	Y (Prop. 3.6)		6	7	Y[1]	N (Prop. 3.3)
	2	6	Y[1]	N (Prop. 3.3)	30	0	15	Y[1]	N (Thm. 3.4)

Table 2: Possible $1 < n \leq 30, \mu, k$ values

4. Truly Complex ETFs from Blocks

Consider matrices of the form

$$\begin{pmatrix} B_0 & B_1 & B_2 \\ B_1^t & D & C \\ B_2^t & C^* & -D \end{pmatrix} \tag{2}$$

where B_0 is a 2×2 matrix with ones on the off diagonal and zeros on the diagonal, B_1 consists of a row of ones followed by a row of negative ones, B_2 is two rows of ones, D is a $\frac{n-2}{2} \times \frac{n-2}{2}$ Seidel matrix with $(\frac{n-2}{4} - 1)$ negative ones in each row, and C is a matrix with entries ± 1 or $\pm i$. Analyzing this pattern, we get the following proposition.

Proposition 4.1. *Let A be a matrix of the form described by Equation* (2)*. Then the following statements are equivalent:*

1. $A^2 = (n-1)I$
2. *C is normal, $CD = DC$, $D^2 + CC^* = (n-1)I - 2J$, and the row and column sums of C are zero.*

Proof. Squaring A yields

$$\begin{pmatrix} B_0^2 + B_1B_1^T + B_2B_2^T & B_0B_1 + B_1D + B_2C^* & B_0B_2 + B_1C - B_2D \\ B_1^TA + DB_1^T + CB_2^T & B_1^T + D^2 + CC^* & B_1^TB_2 + DC - CD \\ B_2^TB_0 + C^*D - DC^* & B_2^TB_1 + C^*D - DC^* & B_2^TB_2 + C^*C + D^2 \end{pmatrix}$$

It is clear that the $(1,1)$-entry of A^2 is the 2×2 identity matrix. Since $B_1^T B_2$ is the zero matrix, it follows that the $(2,3)$-entry of A^2 is the zero matrix if and only if $CD = DC$. The $(2,2)$ and $(3,3)$ entries of A^2 equal $(n-1)I$ if and only if $CC^* = C^*C$ (C is normal) and $D^2 + CC^* = (n-1)I - 2J$. Since $B_0 B_1 = B_1 D$ and $B_0 B_2 = B_2 D$, it follows that the $(1,2)$ and $(1,3)$-blocks of A^2 are the zero matrix if and only if the row and column sums of C are equal to zero.

Proposition 4.1 has led to the construction of new fourth root Seidel matrices with two eigenvalues. Using Proposition 4.1, for each matrix D, we can quickly search for possible matrices C. The blocks D and C are significantly smaller than the overall matrix. This greatly sped up our search. The results of this search are summarized in Table 3. The entry of "by Const." means that a brute force algorithm was used.

n	μ	k	TC?
4	0	2	N (by Const.)
6	0	3	Y (using Prop 4.1)
8	0	4	N (by Const.)
10	0	5	Y (using Prop 4.1)
12	0	6	Y (by Const.)
14	0	7	Y (using Prop 4.1)
16	-2	10	Y (by Const.)
	0	8	Unknown
	2	6	Y (by Const.)
18	0	9	Y (using Prop 4.1)
20	0	10	Unknown

Table 3: Possible $1 < n \leq 20, \mu, k$ values

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[1] B.G. Bodmann and V.I. Paulsen, Frames, graphs and erasures, Linear Algebra Appl. **404** (2005), 118–146.

[2] B.G. Bodmann, V.I. Paulsen, and M. Tomforde, Equiangular Tight Frames from Complex Seidel Matrices Containing Cube Roots of Unity, Linear Algebra Appl. **430** (2009), 396–417.

[3] P.G. Casazza, The art of frame theory, Taiwanese Journal of Mathematics **4** (2000), 129-201, documented as math.FA/9910168 at www.arxiv.org.

[4] The GAP Group, 2006. GAP - Groups, Algorithms, and Programming, Version 4.4. (`http://www.gap-system.org`).

[5] K. Hoffman and R. Kunze, Linear Algebra, Prentice-Hall Inc. Englewood Cliffs, N.J., Second edition, 1971.

[6] R.B. Holmes, Optimal Frames, PhD Thesis, University of Houston, 2003.

[7] R.B. Holmes and V.I. Paulsen, Optimal frames for erasures, Linear Algebra and its Applications **377** (2004), 31–51.

[8] Deepti Kalra, Complex equiangular cyclic frames and erasures, Linear Algebra and its Applications **419** (2006), 373–399.

[9] J. Kovacevic and A. Chebira, Life beyond bases: The advent of frames (Part I), IEEE SP Mag., vol. 24, no. **4**, Jul. 2007, pp. 86-104. Feature article.

[10] T. Strohmer and R. Heath, Grassmannian frames with applications to coding and communications, Appl. Comp. Harm. Anal., **14** (2003), 257-275.

[11] Sustik, Mátyás A. and Tropp, Joel A. and Dhillon, Inderjit S. and Heath, Jr., Robert W., On the existence of equiangular tight frames, Linear Algebra Appl., **426**, (2007), 619-635.

[12] J.H. van Lint and J.J. Seidel, Equilateral point sets in elliptic geometry, Nederl. Akad. Wetensch. Proc. Ser. A 69=Indag. Math., **28**, (1966), 335–348.