Complex Equiangular Tight Frames and Erasures

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Abstract

In this paper we demonstrate that there are distinct differences between real and complex equiangular tight frames (ETFs) with regards to erasures. For example, we prove that there exist arbitrarily large non-trivial complex equiangular tight frames which are robust against three erasures, and that such frames come from a unique class of complex ETFs. In addition, we extend certain results in [1] to complex vector spaces as well as show that other results regarding real ETFs are not valid for complex ETFs.

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1. Introduction

In the last few years the search for equiangular tight frames (ETFs) has become increasingly popular [1, 2, 3, 7, 13, 14, 16, 17]. The main reason for this increased interest is that ETFs minimize the "error" for two erasures in certain communication networks [13, 16].

In this paper we extend a result in [1] regarding real ETFs to complex ETFs. We also demonstrate that there are distinct differences between real and complex ETFs. For example, the real 3-uniform frames correspond precisely to the so-called trivial real ETFs [1]. However, we prove that there exist arbitrarily large non-trivial complex 3-uniform frames, and that such frames come from a unique class of complex ETFs. Consequently, there exist complex ETFs which are also robust against three erasures. Furthermore, we show that there exist only one class of ETFs robust against four erasures, and in some sense this class is "trivial".

The paper is organized as follows: Section 2 outlines the relationship between equiangular tight frames and a certain class of matrices called *Seidel matrices*, and Section 3 includes the results and examples. Readers familiar with the work in [13, 16, 1] may go straight to Section 3.

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2. Preliminaries

It is assumed that the reader is familiar with the basic definitions and theorems of frame theory. Both of the papers [4, 15] are recommended as an introduction to the general theory on frames. For a detailed discussion on ETFs, and much of the motivation behind this paper, the authors further recommend reading [1, 13].

The following definition and theorem are due to Holmes and Paulsen [13].

Definition 2.1. An $n \times n$ self-adjoint matrix Q satisfying $q_{ii} = 0$ and $|q_{ij}| = 1$ for all $i \neq j$ is called a **Seidel matrix**.

Note that some authors refer to a Seidel matrix as a *signature matrix*.

Remark 1. When Q is a real Seidel matrix, $A = 1/2(Q - I + J)$ is the adjacency matrix for a graph. We consider this graph as associated to the frame corresponding to Q.

Theorem 2.2 (Theorem 3.3 of [13]). *Let Q be a Seidel matrix. Then the following are equivalent:*

1. *Q is the Seidel matrix of an equiangular tight frame,*
2. *$Q^2 = (n-1)I + \mu Q$ for some necessarily real number μ,*
3. *Q has exactly two eigenvalues.*

Note that condition (2) in Theorem 2.2 is particularly useful for the computational aspects of constructing a Seidel matrix Q associated with an equiangular tight frame. Furthermore, a Seidel matrix Q satisfying any of the three equivalent conditions in Theorem 2.2 yields several useful parameters. It is shown in [13], if $\lambda_1 < 0 < \lambda_2$ are Q's two eigenvalues, then the parameters n, k, μ, λ_1, and λ_2 satisfy the following properties:

$$\mu = (n - 2k)\sqrt{\frac{n-1}{k(n-k)}} = \lambda_1 + \lambda_2, \qquad k = \frac{n}{2} - \frac{\mu n}{2\sqrt{4(n-1) + \mu^2}} \qquad (1)$$

$$\lambda_1 = -\sqrt{\frac{k(n-1)}{n-k}}, \qquad \lambda_2 = \sqrt{\frac{(n-1)(n-k)}{k}}, \quad n = 1 - \lambda_1\lambda_2.$$

In order to better understand this relationship between a Seidel matrix Q with two distinct eigenvalues and its associated equiangular tight frame, we need the following theorem about finite dimensional frames.

Theorem 2.3. *Let $\mathbb{F}$ denote the field of real or complex numbers. The family $F = \{x_i\}_{i=1}^n \subset \mathbb{F}^k$ is a Parseval frame for $\mathbb{F}^k$ if and only if the analysis operator V associated with F is an isometry.*

For the remainder of this paper it will be assumed that a frame F is a Parseval frame. We will refer to a Parseval frame with n vectors in $\mathbb{F}^k$ as an **(n,k)-frame**. If we consider an element x in $\mathbb{F}^k$ as a column vector, then the rows of the analysis operator V are the adjoints of the frame vectors in F.

In [13, 1], they discuss one way to consider a frame as a code. It is their idea that is the main impetus for the work in this paper. We end this section by outlining this idea, and then show how Seidel matrices arise in the study of (n, k)-frames.

Given a vector x in $\mathbb{F}^k$ and an (n, k)-frame with analysis operator V, consider the vector Vx in $\mathbb{F}^n$ as an encoded version of x, and simply decode Vx by applying V^*. Let E denote the diagonal matrix of m zeros and $n-m$ ones. Thus the vector EVx is just the vector Vx with m-components erased corresponding to the zeros in the diagonal entries of E. It is said that *m-erasures* have occurred during transmission. One way to decode the received vector EVx with m erasures is to again apply V^*. The error in reconstructing x by multiplying EVx on the left by V^* is given by

$$\|x - V^*EV\| = \|V^*(I - E)Vx\| = \|V^*DVx\|$$

where D is the diagonal matrix of m ones and $n - m$ zeros. The operator V^*DV is referred to as the *error operator*. This is only one of several methods possible for reconstructing x. However, it is this particular method which led Bodmann and Paulsen in [1] to introduce the following definition. The quantity in Definition 2.4 represents the maximal norm of an error operator given that some set of m erasures occurs.

Definition 2.4. Let $\mathcal{D}_m$ denote the set of diagonal matrices that have exactly m diagonal entries equal to one and $n - m$ entries equal to zero. Given an (n, k)-frame F, set

$$e_m^\infty(F) := \max\{\|V^*DV\| : D \in \mathcal{D}_m\},$$

where V is the analysis operator of F, and the norm of the matrix is understood to be the operator norm.

An (n, k)-frame F in $\mathbb{F}^k$ where $\|f_i\|$ is a constant for each $i = 1, ..., n$ is commonly referred to as an *equal norm frame* in the current literature. If F has the additional property that $|\langle f_j, f_i \rangle|$ is a constant whenever $i \neq j$, then F is an ETF. For the purposes of this paper, we will refer to *equal norm frames* as **uniform frames** and to *equiangular tight frames* as **2-uniform frames** as in [13]. Both uniform and 2-uniform frames are important since an (n, k)-frame F minimizes the quantity $e_1^\infty(F)$ if and only if F is a uniform frame [5], and minimizes the quantity $e_2^\infty(F)$ if and only if F is a 2-uniform frame [13]. Indeed, uniform and 2-uniform frames are robust against one and two erasures respectively. In Section 3 we show that there exist a class of complex (n, k)-frames which are robust against one, two and three erasures. Furthermore, we show that this is the only such class of frames.

We end this section by showing the connection between 2-uniform frames (or ETFs) and Seidel matrices. Suppose F is a 2-uniform frame, and that V is the associated analysis operator for F. It is easy to show that an (n, k)-frame F is a uniform frame if and only if $\|f_i\| = \sqrt{\frac{k}{n}}$ for each $i = 1, ..., n$. It follows that

$$VV^* = \frac{k}{n}I + c_{n,k}Q$$

where Q is a Seidel matrix and $c_{n,k} = \sqrt{\frac{k(n-k)}{n^2(n-1)}} = |\langle f_j, f_i \rangle|$. It is worth noting that each $n \times n$ Seidel matrix produces a set of equiangular lines in $\mathbb{C}^k$ for some $k < n$. However, the vectors corresponding to this set of equiangular lines do not necessarily span $\mathbb{C}^k$, and consequently they may not form a frame for $\mathbb{C}^k$. This is precisely why Theorem 2.2 is important, it provides necessary and sufficient conditions in order for a Seidel matrix (real or complex) to produce a 2-uniform frame (ETF).

3. Results

Definition 3.1. Let F be an (n, k)-frame in $\mathbb{F}^k$. We will call F an **m-uniform frame** provided that $\|V^*DV\|$ is a constant for each D in $\mathcal{D}_m$. F is called a **completely m-uniform frame**, denoted m_c**-uniform frame**, if F is an ℓ-uniform frame for each $\ell = 1, ..., m$.

Note, that there is a distinction between a 2-*uniform frame* F in the above definition and what the authors in [13, 1] refer to as a 2-*uniform frame*. Namely, that a 2-uniform frame in [13, 1] is what we refer to as a 2_c-uniform frame.

Along with introducing the error operator stated in Definition 2.4, the authors in [1] developed error estimates of this operator. Their key result for these estimates is Theorem 5.3, which we restate here.

Theorem 3.2 (Theorem 5.3 of [1]). *Let F be a real 2-uniform (n, k)-frame. Then $e_m^\infty \leq k/n + (m-1)c_{n,k}$ with equality if and only if a graph associated with F contains an induced subgraph on m vertices that is complete bipartite.*

The following proposition summarizes results about real 3_c-uniform frames which follow as corollaries to Theorem 5.3 in [1].

Proposition 3.3. *Let F and G be real 2_c-uniform (n,k)-frames.*

1. *The graph associated with F either contains an induced complete bipartite graph on 3 vertices or it is switching equivalent to the complete graph on n vertices. Consequently, if $k < n - 1$, then $e_3^\infty(F) = \frac{k}{n} + 2c_{n,k}$.*
2. *$e_3^\infty(F) = e_3^\infty(G)$.*
3. *The trivial 2_c-uniform (n,k)-frames, corresponding to $k = 1$ and $k = n-1$, are 3-uniform. Conversely, if F is a real 3_c-uniform (n,k)-frame, then either $k = 1$ or $k = n - 1$ and it is equivalent to the corresponding trivial frame.*

In [1], the authors used the connection between real Seidel matrices and graphs to prove Theorem 3.2 and the results listed in Proposition 3.3. They extended this connection to Seidel matrices containing third roots of unity using directed graphs in [2]. Unfortunately, there is no obvious extension of this idea to connect arbitrary complex Seidel matrices with a currently known class of graphs. However, the fact that these known proofs of the real do not extend to the complex case does not mean these statements do not hold. In particular, we are able to recover an analog for Theorem 5.3 and one of its corollaries. Furthermore, we provide counterexamples for the other two corollaries.

The following proposition is the key ingredient in determining an upper bound for $e_m^\infty(F)$ as well as when the upper bound is saturated. The real case of Proposition 3.4 below is part of the proof of Theorem 5.3 in [1].

Proposition 3.4. *If Q is a Seidel adjacency matrix , then $\|Q\|$ is at most $n-1$. Moreover, $\|Q\| = n - 1$ if and only if $Q = J - I$.*

PROOF. First note that the largest eigenvalue of J_n, the matrix of all ones, is n. For any vector x in $\mathbb{C}^n$ and any S, taking the moduli of all their entries can only increase the value of the expression

$$\frac{|\langle (I_n + Q)x, x \rangle|}{\|x\|^2}. \tag{2}$$

Since $I_n + Q$ is a Hermitian matrix $\|I_n + Q\|$ is the maximum of the moduli of the eigenvalues of $I_n + Q$. Let x be an eigenvector of $I_n + Q$ corresponding to the eigenvalue, λ, of largest modulus, and let $\overline{x} = (|x_1|, ..., |x_n|)$. It follows that:

$$\|I_n + Q\| = |\lambda| = \frac{|\langle (I_n + Q)x, x \rangle|}{\|x\|^2} \leq \frac{|\langle J_n \overline{x}, \overline{x} \rangle|}{\|x\|^2} \leq \frac{\|J_n \overline{x}\|\|x\|}{\|x\|^2} \leq n.$$

Hence, $\|Q\|$ is at most $n - 1$.

Since n is the largest eigenvalue of matrix J it follows that $n - 1$ is largest eigenvalue of $Q = J - I$. In addition, the expression in (2) can only increase. Thus, if Q does not equal $J - I$, then $\|Q\| < n - 1$.

Let Q_m denote a compression of Q to m rows and m columns. We say two Seidel matrices Q and S are *switching equivalent* if there exists a permutation matrix P and a diagonal matrix D whose diagonal entries have modulus 1 such that $Q = PDSD^{-1}P^{-1}$.

Corollary 3.5. *Let F be 2_c-uniform (n,k)-frame (real or complex) and let Q be the associated Seidel matrix of the corresponding projection VV^*. Then*

$$e_m^\infty(F) \leq \frac{k}{n} + (m-1)c_{n,k} \tag{3}$$

with equality if and only if there is a Q_m switching equivalent to $J_m - I_m$.

PROOF. Let F be an equiangular (n,k)-frame and V be the corresponding analysis operator for F. Since VV^* is a positive operator, its compression $(VV^*)_m$, where $1 \leq m \leq n$, to the rows and columns where D has 1's, is also a positive operator. Thus, determining the norm of $\|V^*DV\| = \|DVV^*D\|$ is equivalent to finding the largest eigenvalue of $(VV^*)_m$. We further reduce this problem to finding the largest eigenvalue of Q_m, where $(VV^*)_m = \frac{k}{n}I + c_{n,k}Q_m$. By Proposition 3.4,

$$\|DVV^*D\| = \|D(\frac{k}{n}I_n + c_{n,k}Q)D\| = \|\frac{k}{n}I_m + c_{n,k}Q_m\| \leq \frac{k}{n} + (m-1)c_{n,k}$$

with equality if and only if $Q = J - I$.

Remark 2. The above corollary is the complex version of Theorem 5.3 in [1]. In [1], 2_c-uniform (n,k)-frames for which $\|DVV^*D\|$ is a constant for every D in $\mathcal{D}_3$ are called *3-uniform*, or in the terminology of this paper 3_c-uniform.

Example 3.6 below shows that there are complex 2_c-uniform (n,k)-frames, say F and G, for which $e_3^\infty(F) \neq e_3^\infty(G)$ which violates parts (1) and (2) of Proposition 3.3.

Example 3.6. *Let F and G be the complex 2_c-uniform $(9,3)$-frames corresponding to the Seidel matrices*

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & -1 & \omega^5 & \omega^5 & \omega^5 & \omega & \omega & \omega \\ 1 & -1 & 0 & \omega & \omega & \omega & \omega^5 & \omega^5 & \omega^5 \\ 1 & \omega & \omega^5 & 0 & \omega^5 & \omega & -1 & \omega^5 & \omega \\ 1 & \omega & \omega^5 & \omega & 0 & \omega^5 & \omega^5 & \omega & -1 \\ 1 & \omega & \omega^5 & \omega^5 & \omega & 0 & \omega & -1 & \omega^5 \\ 1 & \omega^5 & \omega & -1 & \omega & \omega^5 & 0 & \omega & \omega^5 \\ 1 & \omega^5 & \omega & \omega & \omega^5 & -1 & \omega^5 & 0 & \omega \\ 1 & \omega^5 & \omega & \omega^5 & -1 & \omega & \omega & \omega^5 & 0 \end{bmatrix}$$

and

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & -1 & \omega^5 & \omega^5 & \omega^5 & \omega & \omega & \omega \\ 1 & -1 & 0 & \omega & \omega & \omega & \omega^5 & \omega^5 & \omega^5 \\ 1 & \omega & \omega^5 & 0 & \omega^5 & \omega & 1 & \omega^4 & \omega^2 \\ 1 & \omega & \omega^5 & \omega & 0 & \omega^5 & \omega^2 & 1 & \omega^4 \\ 1 & \omega & \omega^5 & \omega^5 & \omega & 0 & \omega^4 & \omega^2 & 1 \\ 1 & \omega^5 & \omega & 1 & \omega^4 & \omega^2 & 0 & \omega^5 & \omega \\ 1 & \omega^5 & \omega & \omega^2 & 1 & \omega^4 & \omega & 0 & \omega^5 \\ 1 & \omega^5 & \omega & \omega^4 & \omega^2 & 1 & \omega^5 & \omega & 0 \end{bmatrix}$$

respectively, where ω is a primitive 6th root of unity. By computation, we get $e_3^\infty(F) \approx .6465$ which is strictly less than $\frac{k}{n} + 2c_{n,k} = \frac{2}{3}$ disproving part (1) of Proposition 3.3. Since $e_3^\infty(G) \approx \frac{2}{3}$, we also see that part (2) of Proposition 3.3 fails to hold for complex matrices.

Part (3) of Proposition 3.3 states that the only real 3_c-uniform (n,k)-frames are the trivial (n, k)-frames. However, the following example shows that in the complex case there exist non-trivial 3_c-uniform frames.

Example 3.7. *Let F and G be the complex 2_c-uniform frames corresponding to the Seidel matrices*

$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & -i & i \\ 1 & i & 0 & -i \\ 1 & -i & i & 0 \end{bmatrix}$$

and

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & -i & -i & -i & i & i & i \\ 1 & i & 0 & -i & i & -i & -i & i \\ 1 & i & i & 0 & -i & -i & i & -i \\ 1 & i & -i & i & 0 & i & -i & -i \\ 1 & -i & i & i & -i & 0 & -i & i \\ 1 & -i & i & -i & i & i & 0 & -i \\ 1 & -i & -i & i & i & -i & i & 0 \end{bmatrix}$$

respectively. These frames are both 3_c-uniform and neither of them is a trivial $(n, 1)$ or $(n, n - 1)$-frame.

The 2_c-uniform frames corresponding to the Seidel matrices in Example 3.7 come from real skew-symmetric matrices with two distinct eigenvalues. A more detailed discussion of 2_c-uniform frames which arise from such matrices can be found in [7]. The following theorem shows that all 2_c-uniform frames which arise from a real skew-symmetric matrix with two distinct eigenvalues are 3_c-uniform.

Theorem 3.8. *Let A be real skew-symmetric matrix with two distinct eigenvalues and entries $a_{i,j} = \pm 1$ when $i \neq j$ and 0 otherwise. The frame corresponding to the Seidel matrix $Q = iA$ is 3_c-uniform.*

PROOF. By Proposition 3.1 of [7], the standard form of Q has entries

$$q_{i,j} = \begin{cases} \pm i, & \text{if } 1 < i, \, 1 < j, \text{ and } i \neq j; \\ 0, & \text{if } i = j; \\ 1, & \text{otherwise} \end{cases}$$

Thus, every compression of Q to three rows and three columns is either of the form

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & i \\ 1 & -i & 0 \end{bmatrix} \text{ or } \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & -i \\ 1 & i & 0 \end{bmatrix}$$

Consequently, $\|V^*DV\|$ is a constant for all D in $\mathcal{D}_3$ from which the result follows.

Corollary 3.9. *There exist 3_c-uniform (n, k)-frames for arbitrarily large values of n.*

The proof of Corollary 3.9 follows from Proposition 3.6 in [7]. While Theorem 3.8 shows that arbitrarily large non-trivial 3_c-uniform frames exist, there is still the question of "Are there non-trivial 3_c-uniform frames which come from Seidel matrices with entries other than i or $-i$?". Theorem 3.10 answers this question. Furthermore, it distinguishes the complex case from the real case, and is the complex analog of Part (3) of Proposition 3.3.

Theorem 3.10. *The trivial 2_c-uniform frames, corresponding to $k = 1$ or $k = n - 1$, are 3_c-uniform. In addition, F is a non-trivial 3_c-uniform frame if and only if F is a 2_c-uniform frame arising from a real skew-symmetric matrix A with two distinct eigenvalues and entries $a_{i,j} = \pm 1$ when $i \neq j$ and 0 otherwise.*

The following two lemmas will be used to prove Theorem 3.10.

Lemma 3.11. *Suppose $1 \leq \lambda \leq \gamma \leq 3$ be the largest roots of the polynomials $x^3 - 3x^2 + 2 - 2\cos(\alpha)$ and $x^3 - 3x^2 + 2 - 2\cos(\beta)$, respectively. Then $\alpha \geq \beta$. Furthermore, when $0 \leq \alpha \leq \beta \leq \pi$, equality holds if and only if $\lambda = \gamma$.*

PROOF. By assumption,

$$\lambda - 3 \leq \gamma - 3$$

which gives

$$\lambda^2(\lambda - 3) \leq \gamma^2(\gamma - 3).$$

Combining this with the polynomials we get

$$2 - 2\cos(\alpha) \leq 2 - 2\cos(\beta),$$

so $\cos(\alpha) \leq \cos(\beta)$ and $\alpha \geq \beta$.

Lemma 3.12. *Suppose F is a 3_c-uniform frame with corresponding Seidel matrix Q. Then the entries q_{ij} of Q are of the form ω or $\bar{\omega}$ when $1 < i$, $1 < j$ and $i \neq j$, for some fixed complex number ω with modulus 1.*

PROOF. Let M and N be two 3×3 compressions of the Seidel matrix Q corresponding to the 3-uniform frame F. Since conjugating by an invertible matrix preserves eigenvalues, we can change M and N to be written as

$$M = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & \alpha \\ 1 & \bar{\alpha} & 0 \end{bmatrix}$$

and

$$N = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & \beta \\ 1 & \bar{\beta} & 0 \end{bmatrix}$$

where α and β are complex numbers with modulus 1. The characteristic polynomials of $M+I_3$ and $N+I_3$ are $x^3-3x^2+2-2\cos(\alpha)$ and $x^3-3x^2+2-2\cos(\beta)$. Polynomials of this form are discussed in Proposition 3.11. Since the norms of all 3×3 compressions are equal, α and β must be equal or conjugates.

The reverse direction is clear.

PROOF (OF THM. 3.10). In [1], they observe that the trivial real 2_c-uniform frames are 3_c-uniform.

Without loss of generality assume that the Seidel matrix Q associated with F is in standard form. In the complex case, if F is 3_c-uniform, then Lemma 3.12 forces the off diagonal entries of the $(n-1) \times (n-1)$ compression formed by removing the first row and column of Q to be either of the form ω or $\bar{\omega}$ where $|\omega|=1$.

Suppose that $i,j>1$, $i \neq j$, and the (i,j)-entry of Q is ω. Using the fact that $Q^2=(n-1)I+\mu Q$ it follows that

$$\mu\omega = m_1 + m_2\omega^2 + m_3\bar{\omega}^2$$

where m_1, m_2, and m_3 are positive integers. If $m_1 = m_2$, then $m_1 2\mathrm{Re}(\omega)+\mu = m_3\bar{\omega}^3$ which forces μ to be complex. If $m_1 > m_2$, then

$$m_2 2\mathrm{Re}(\omega) + \mu = (m_1 - m_2)\bar{\omega} + m_3\bar{\omega}^3. \tag{4}$$

Clearly if $m_1 - m_2 \neq m_3$, the right-hand side of (4) is complex. On the other hand if $m_1 - m_2 = m_3$ and $\omega = \mathrm{e}^{i\theta}$ for some $0 \leq \theta < 2\pi$, then $\mathrm{e}^{i\theta} + \mathrm{e}^{i3\theta}$ must be a real number. But this means that $\sin(\theta) + \sin(3\theta) = 0$ which occurs if and only if ω is a fourth root of unity as desired.

Theorem 3.13. *The only nontrivial 4_c-uniform frames are the ones in the equivalence class given by the 4×4 Seidel matrix in Example 3.7 previously mentioned.*

PROOF. By Theorem 3.10 we know that a nontrivial 3_c-uniform frame corresponds to Seidel matrix Q with entries

$$Q_{jm} = \begin{cases} 0 & \text{if } m = j, \\ 1 & \text{if } m \neq j \text{ and } m = 1 \text{ or } j = 1, \\ \pm i & \text{otherwise.} \end{cases}$$

The proof of Theorem 3.10 shows that the sum of the entries in each row and column of q, other than the first, is 1.

Suppose Q is an $n \times n$ matrix, then we use Q to describe an edge coloring of the complete graph K_{n-1}. Label the vertices by the integers $2, \ldots, n$. Color the edge from vertex j to vertex m red if $q_{jm} = -i$ and blue otherwise. It is well known, see [6, 10], that the Ramsey number $r(3,3) = 6$. With our interpretation of Q giving a coloring, when $n \geq 7$ our coloring of K_{n-1} contains

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a monochromatic triangle. The labels of the vertices of this triangle along with 1 give us a 4×4 compression of Q of the form

$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & i & i \\ 1 & -i & 0 & i \\ 1 & -i & -i & 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & -i & -i \\ 1 & i & 0 & -i \\ 1 & i & i & 0 \end{bmatrix}.$$

Without loss of generality, assume that the first of the possible compressions above is the top left corner of Q. Since the row sums of Q are 1, there is a column of Q such that $q_{j2} = -i$ and $q_{j3} = i$. With this, the 4×4 compression using the rows and columns $\{1, 2, 3, j\}$ has the form

$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & i & -i \\ 1 & -i & 0 & i \\ 1 & i & -i & 0 \end{bmatrix}.$$

These two compressions have different norms, so Q is not 4-uniform. A similar argument works for the other possible compression above.

The cases where $n < 7$ have been checked computationally.

In [1], the authors showed that the only real 3_c-uniform frames are the trivial $(n, n - 1)$ and $(n, 1)$ frames. Theorem 3.10 extends this classification of 3_c-uniform frames to the complex case. In addition to the real 3_c-uniform frames, we add a new class, in particular the frames derived from real skew-symmetric matrices with exactly two eigenvalues. Theorem 3.13 takes this classification one step farther to show that the only real or complex 4_c-uniform frames are the trivial frames and one more, Example 3.7, which is 4_c-uniform for the trivial reason that it has only one 4×4 compression.

We end the paper by interpreting these results geometrically. A uniform (n, k)-frame yields a set of n-vectors in $\mathbb{R}^k$ (or $\mathbb{C}^k$) which have equal lengths. Another way to interpret 2-uniform (n, k)-frames (or equivalently ETFs) is that the area of the parallelogram formed by any two distinct vectors from such a frame is a constant. Intuitively, it would seem that the volume of the parallelepiped formed by choosing any three distinct vectors from a 2-uniform (n, k)-frame should be a constant. However, this is not true in general. In the real case, this is true if and only if the frame is trivial [1], i.e., either an $(n, 1)$ or $(n, n - 1)$ frame. Similarly, we have proven that in the complex case, the volume of the parallelepiped formed by choosing any three distinct vectors from a 2-uniform (n, k)-frame is a constant if and only if the Seidel matrix associated with the frame comes from a real skew-symmetric with exactly two eigenvalues.

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