Show all work.

1. Show that the limit does not exist: \( \lim_{(x,y) \to (0,0)} \frac{xy}{x^2 + y^2} \).

\[
\begin{align*}
\text{Let } y &= mx \\
\lim_{(x,y) \to (0,0)} \frac{xy}{x^2 + y^2} &= \lim_{y \to mx} \frac{xy}{x^2 + y^2} \\
&= \frac{m}{1 + m^2}.
\end{align*}
\]

If \( m = 0 \) then the limit is 0, and if \( m = 1 \) then the limit is \( \frac{1}{2} \).

So the limit does not exist.

2. Verify that the conclusion of Clairaut’s Theorem holds for the function \( f(x,y) = \ln \sqrt{x^2 + y^2} \). That is, show that \( f_{xy} = f_{yx} \). Hint: \( \ln(x^n) = n \ln(x) \).

\[
\begin{align*}
f(x,y) &= \frac{1}{2} \ln(x^2 + y^2) \\
f_x &= \frac{1}{x^2 + y^2} \cdot 2x = \frac{x}{x^2 + y^2} \\
f_{xy} &= \frac{(x^2 + y^2)(0) - (x)(2y)}{(x^2 + y^2)^2} = -\frac{2xy}{(x^2 + y^2)^2} \\
f_y &= \frac{1}{2} \cdot \frac{1}{x^2 + y^2} \cdot 2y = \frac{y}{x^2 + y^2} \\
f_{yx} &= \frac{(x^2 + y^2)(0) - y(2x)}{(x^2 + y^2)^2} = -\frac{2xy}{(x^2 + y^2)^2} = f_{xy}.
\end{align*}
\]
3. Find the equation of the tangent plane to the elliptic paraboloid \( z = f(x, y) = 2x^2 + y^2 \)
at \((x, y) = (1, 1)\) and use it to approximate \( f(1.1, 0.95) \).

\[
\begin{align*}
  f_x &= 4x \implies f_x(1,1) = 4 \\
  f_y &= 2y \implies f_y(1,1) = 2 \\
  f(1,1) &= 2(1)^2 + 1^2 = 3 \\
  z - 3 &= f_x(x_0, y_0)(x-x_0) + f_y(x_0, y_0)(y-y_0) \\
\end{align*}
\]

\[
\Rightarrow z - 3 = 4(x-1) + 2(y-1) \\
\boxed{z = 3 + 4(x-1) + 2(y-1)} \text{ is a tangent plane}
\]

Next,
\[
f(1.1, 0.95) \approx 3 + 4(1.1-1) + 2(0.95-1)
\]

Using the tangent plane,
\[
= 3 + 4(0.1) + 2(-0.05) \\
= 3 + 0.4 - 0.1 \\
= 3.3 \text{ approximation}
\]
4. State the relevant chain rule formula and use it to find \( \frac{\partial z}{\partial t} \) if \( z = \tan^{-1}(2x+y), x = s^2t \) and \( y = s \ln t \).

\[
\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt} \quad \text{(relevant formula)}
\]

\[
= \left[ \frac{1}{1 + (2x+y)^2} \right] \cdot 2s^2 + \left[ \frac{1}{1 + (2x+y)^2} \right] \cdot \frac{s}{t} \nabla^2 \frac{\partial z}{\partial t}
\]

\[
= \frac{2s^2}{1 + (2x+y)^2} + \frac{s}{t} \left[ 1 + (2x+y)^2 \right]
\]

\[
= \frac{2s^2 t + s}{t \left[ 1 + (2x+y)^2 \right]}
\]
5. Find the directional derivative of \( f(x, y) = \frac{x}{y} \) at the point \((6, -2)\) in the direction of \( v = (4, -3)\). What direction will give the greatest decrease of \( f \) at that point?

\[
\vec{V} = (4, -3) \quad \text{so} \quad \vec{U} = \left< \frac{4}{5}, -\frac{3}{5} \right> \quad \text{unit vector}
\]

\[
\nabla f(x, y) = \left< \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right> = \left< \frac{1}{y}, -\frac{x}{y^2} \right>
\]

\[
\nabla f(6, -2) = \left< -\frac{1}{2}, \frac{6}{4} \right> = \left< -\frac{1}{2}, -\frac{3}{2} \right>
\]

and \( \vec{D}_{\vec{U}} f(6, -2) = \vec{U} \cdot \nabla f(6, -2) \)

\[
= \left< \frac{4}{5}, -\frac{3}{5} \right> \cdot \left< -\frac{1}{2}, -\frac{3}{2} \right>
\]

\[
= -\frac{4}{10} + \frac{9}{10} = \frac{5}{10} = \frac{1}{2}
\]

Directional derivative.

The direction of greatest decrease is \( v \) if

\[
\text{opposite the gradient,}
\]

so in the direction of \( -\nabla f(6, -2) \)

\[
= \left< \frac{1}{2}, \frac{3}{2} \right>.
\]
6. Find the location of the local maximums, local minimums and saddle points of the function \( f(x, y) = x^4 + 2y^2 - 4xy. \)

\[
f_x = 4x^3 - 4y = 0
\]

\[
f_y = 4y - 4x = 0 \implies y = x
\]

Plugging \( y = x \) into \( f_x = 0 \) gives

\[
yx^3 - 4x = 0 \implies y(x^3 - 1) = 0
\]

\[
\implies x \in 0, \pm 1.
\]

Since \( y = x \), the critical points are \((0,0), (-1, -1) \) and \((1, 1)\).

\[
D(x, y) = f_{xx} f_{yy} - f_{xy}^2 = (12x^2)(4) - (-4)^2
\]

\[
= 48x^2 - 16
\]

So \( D(0, 0) = -16 \) \( \implies \) saddle point at \((0, 0)\)

\( D(1, 1) = 48 - 16 = 32 > 0 \) and \( f_{xx} = 12x^2 > 0 \) at \((1, 1)\)

So local min at \((1, 1)\)

\( D(-1, -1) = 32 > 0 \) and \( f_{xx} (-1, -1) = 12 > 0 \)

So local min at \((-1, -1)\) also.
7. Find the absolute maximum and minimum values of the function \( f(x, y) = x^2 + y^2 - x - y + 1 \) on the unit square \( S = \{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq 1\} \).

Critical points:
\[
\begin{align*}
\frac{\partial f}{\partial x} &= 2x - 1 = 0 \\
\frac{\partial f}{\partial y} &= 2y - 1 = 0
\end{align*}
\]

\( \Rightarrow \) \( x = \frac{1}{2}, \ y = \frac{1}{2} \)

\( \Rightarrow (\frac{1}{2}, \frac{1}{2}) \) is the only c.p.

\[
f\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{1}{4} + \frac{1}{4} - \frac{1}{2} - \frac{1}{2} + 1 = \frac{1}{2}
\]

On \( S_1 : \ y = 0, \ 0 \leq x \leq 1 \)

\( f(x, 0) = x^2 - x + 1 = g(x) \) \( \Rightarrow \)

\[
g'(x) = 2x - 1 = 0 \Rightarrow x = \frac{1}{2}
\]

\( g(0) = 1, \quad g\left(\frac{1}{2}\right) = \frac{1}{4} - \frac{1}{2} + 1 = \frac{3}{4}, \quad g(1) = 1 \)

On \( S_2 : \ x = 0, \ 0 \leq y \leq 1 \)

\( f(0, y) = y^2 - y + 1 = g(y) \) (same as side 1)

On \( S_3 : \ y = 1, \ 0 \leq x \leq 1 \)

\( f(x, 1) = x^2 + 1 - x - 1 + 1 = x^2 - x + 1 \) (same as side 1)

On \( S_4 : \ x = 1, \ 0 \leq y \leq 1 \)

\( f(1, y) = 1 + y^2 - 1 - y + 1 = y^2 - y + 1 \) (same as side 1)

\[\text{So also max} = 1 \text{ at } (0, 0), (1, 0), (0, 1) \text{ and } (1, 1)\]

\[\text{also min} = \frac{1}{2} \text{ at } \left(\frac{1}{2}, \frac{1}{2}\right)\]