1. Write an equation that expresses the continuity of the given function at the given value.

a) \( \sqrt{x} \) at \( x = 3 \)
\[
\lim_{x \to 3} \sqrt{x} = \sqrt{3}
\]

b) \( \ln(x) \) at \( x = e \)
\[
\lim_{x \to e} \ln(x) = \ln(e)
\]

c) \( \cos(x) \) at \( x = \pi \)
\[
\lim_{x \to \pi} \cos(x) = \cos(\pi)
\]

2. Where is the function not continuous (or is it continuous everywhere)?

a) \( f(x) = \frac{x + 1}{x^2 - x - 12} \)

Not continuous when \( x^2 - x - 12 = 0 \)
\[
(x - 4)(x + 3) = 0
\]
\[
x = 4, -3
\]

\( \therefore \) \( f(x) \) is not cont. at \( x = 3 \) and at \( x = 4 \).

b) \( f(x) = \frac{x + 1}{x^4 + 2} \)
Continuous everywhere since \( x^4 + 2 \neq 0 \) for any \( x \).

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c) \( f(x) = \frac{x + 1}{1 - e^x} \)

\[ -e^{x - 0} = e^x = 1 \Rightarrow x = 0 \]

Not cont. at \( x = 0 \).

d) \( f(x) = \frac{\tan(x)}{x} \)
\[
\frac{\sin x}{x \cos x}
\]

Not cont. when \( x = 0 \) and when \( \cos x = 0 \)
\[ \cos x = 0 \Rightarrow x = \frac{\pi}{2} + k\pi \text{ when } k \text{ is any integer.} \]
3. Use the definition of continuity to explain why each of the functions graphed below is not continuous at $x = 2$. Also, what type of discontinuity is it?

**a)**

\[ \lim_{x \to 2} f(x) \text{ exists and } f(2) \text{ is defined but } \lim_{x \to 2} f(x) \neq f(2) . \]

Remove Discontinuity

**b)**

\[ \lim_{x \to 2^-} f(x) \neq \lim_{x \to 2^+} f(x) \]

So, \( \lim_{x \to 2} f(x) \) DOES NOT EXIST.

Jump Dis.

**c)**

\[ \lim_{x \to 2^-} f(x) = +\infty \neq f(2) \]

\[ \lim_{x \to 2^+} f(x) = -\infty \neq f(2) \]

Infinite Discontinuity
4. Is \( f(x) \) continuous at \( x = 2 \)? Justify your answer.

\[
f(x) = \begin{cases} 
  x + 3, & \text{if } x \leq 2 \\
  x^2 - 2, & \text{if } x > 2
\end{cases}
\]

\[
limes_{x \to 2^-} f(x) = limes_{x \to 2^-} (x + 3) = 2 + 3 = 5
\]

\[
limes_{x \to 2^+} f(x) = limes_{x \to 2^+} (x^2 - 2) = 4 - 2 = 2
\]

End

\[
limes_{x \to 2} f(x) \text{ DNE and } f(x) \text{ is not continuous at } x = 2.
\]

5. Find the value of \( k \) that makes \( f(x) \) continuous at \( x = 2 \).

\[
f(x) = \begin{cases} 
  kx - 1, & \text{if } x \leq 2 \\
  x^2 + k, & \text{if } x > 2
\end{cases}
\]

Need 1-sided limits to be the same.

\[
limes_{x \to 2^-} f(x) = limes_{x \to 2^-} (kx - 1) = 2k - 1
\]

\[
limes_{x \to 2^+} f(x) = x^2 + k = 4 + k
\]

\[
\therefore \text{ fn k: } 2k - 1 = 4 + k
\]

\[
\boxed{k = 5}
\]
6. Use your knowledge of familiar functions and the properties of continuous functions to explain why each of the following functions is continuous everywhere.

a) \( f(x) = \sin(x) + \cos(x) \)

\( \sin(x) \) and \( \cos(x) \) are continuous everywhere and the sum of continuous functions is continuous.

b) \( g(x) = x^2 - e^x \)

\( x^2 \) (polynomial) and \( e^x \) (exponential) are continuous everywhere and the difference of continuous functions is continuous.

c) \( h(x) = x \tan^{-1}(x) \)

\( x \) (polynomial) and \( \tan^{-1}(x) \) are continuous everywhere and the product of continuous functions is continuous.

d) \( k(x) = \sin(3x) \)

\( \sin(x) \) and \( 3x \) (polynomial) are continuous everywhere and the composition of continuous functions is continuous.
7. Use the Intermediate Value Theorem to show that the equation has a solution in the indicated interval.

a) \(2x - x^3 = -1\) on the interval \((1, 2)\).

\(f(x) = 2x - x^3\) is continuous everywhere since it is a polynomial.

\[f(1) = 2 - 1 = 1 > -1\]

and \(f(2) = 4 - 8 = -4 < -1\).

So \(f(2) < -1 < f(1)\).

By IVT \(f(x) = 2x - x^3 = -1\) for some value of \(x\) between \(x=1\) and \(x=2\).

b) \(e^x = 3x\) on the interval \((0, 1)\).

\(f(x) = e^x - 3x\) is continuous everywhere.

\[f(0) = e^0 - 3(0) = 1 - 0 = 1 > 0\]

and \(f(1) = e^1 - 3(1) = e - 3 < 0\) \((e \approx 2.7)\)

So \(f(1) < 0 < f(0)\).

By IVT \(f(x) = e^x - 3x = 0\) for some value of \(x\) between \(x=0\) and \(x=1\).

(Not \(e^x = 3x\) is equivalent to \(e^x - 3x = 0\))
8. Find the derivative, \( f'(x) \), and the equation of the tangent line to the given function at the given point.

a) \( f(x) = \sqrt{x+1} \) at \( x = 3 \).

\[
f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\sqrt{x+h+1} - \sqrt{x+1}}{h}
\]

\[
= \lim_{h \to 0} \frac{\sqrt{x+h+1} - \sqrt{x+1}}{h} \cdot \frac{\sqrt{x+h+1} + \sqrt{x+1}}{\sqrt{x+h+1} + \sqrt{x+1}} = \lim_{h \to 0} \frac{(x+h+1) - (x+1)}{h(\sqrt{x+h+1} + \sqrt{x+1})}
\]

\[
= \lim_{h \to 0} \frac{1}{\sqrt{x+h+1} + \sqrt{x+1}} = \frac{1}{2\sqrt{x+1}}
\]

**Tangent line**

\[y - 2 = \frac{1}{4}(x-3) \Rightarrow y = \frac{1}{4}x + \frac{5}{4}\]

b) \( f(x) = \frac{1}{x} \) at \( x = 2 \).

\[
f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} = \lim_{h \to 0} \frac{x - (x+h)}{x(x+h)}
\]

\[
= \lim_{h \to 0} \frac{-1}{x(x+h)} \cdot \frac{1}{h} = -\frac{1}{x^2}
\]

**Tangent line**  \( x = 2 \) \( \Rightarrow \) \[y = \frac{1}{x} = \frac{1}{2} \Rightarrow \) point \( (2, \frac{1}{2}) \)

\( \text{slope} = f'(2) = -\frac{1}{2^2} = -\frac{1}{4} \Rightarrow \) line is \( y - y_0 = m(x-x_0) \)

\[\Rightarrow y - \frac{1}{2} = -\frac{1}{4}(x-2) \Rightarrow y = -\frac{1}{4}x + \frac{3}{2}\]